Bilevel Problems, MPCCs, and Multi-Leader-Follower Games

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- Professor in Applied Mathematics at Univ. of Perpignan

Perpignan, France

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  - Quasiconvex optimization

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- Research lab.: PROMES (CNRS)

What do I work on?

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Optimization /Math. programming

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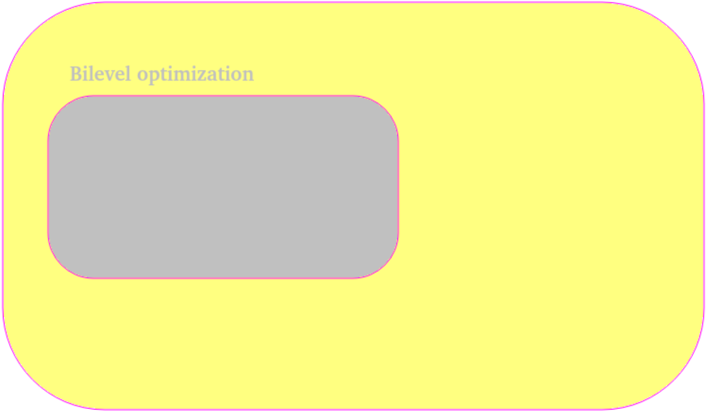

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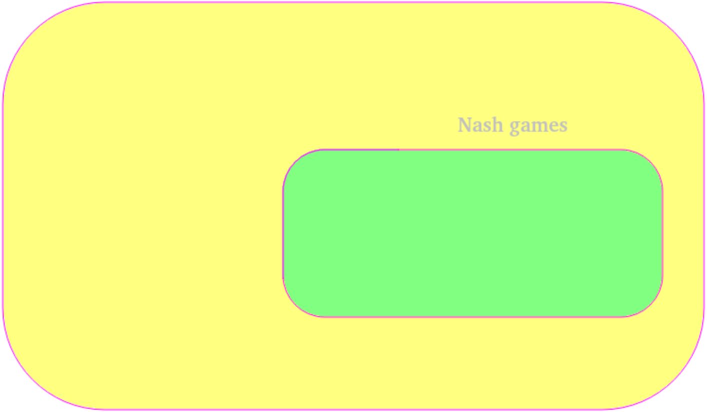

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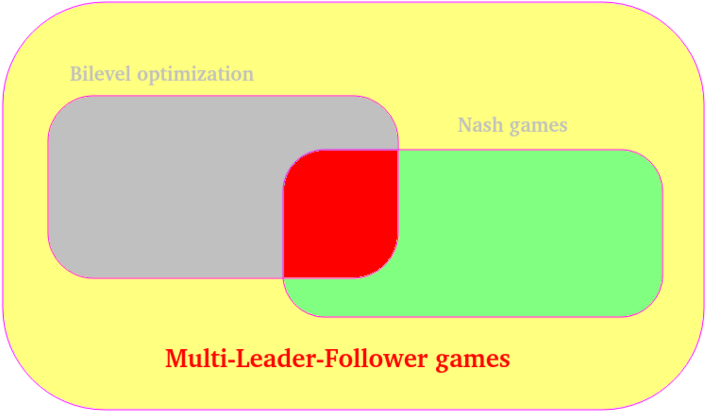
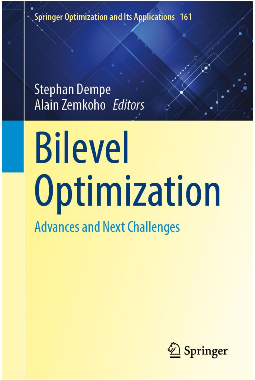


an advertisement

A short state of art on Multi-Leader-Follower games, D.A. and A. Svensson, in a book dedicated to Stackelberg, editors A. Zemkoho and S. Dempe, Springer Ed. (2019)



Bilevel: some general comments

BL: a first definition

A Bilevel Problem consists in an **upper-level/leader's problem**

$$\begin{array}{rl} \text{“}\min_{x\in\mathbb{R}^n}\text{”} & F(x,y) \\ s.t. & \left\{ \begin{array}{l} x \in X \\ y \in S(x) \end{array} \right. \end{array}$$

where $\emptyset \neq X \subset \mathbb{R}^n$ and $S(x)$ stands for the solution set of its **lower-level/follower's problem**

$$\begin{array}{rl} \min_{y\in\mathbb{R}^m} & f(x,y) \\ s.t & g(x,y) \leq 0 \end{array}$$

A trivial example

Consider the following simple bilevel problem

$$\text{“}\min_{x\in\mathbb{R}}\text{”} \quad x$$
$$s.t. \quad \begin{cases} x \in [-1,1] \\ y \in S(x) \end{cases}$$

with $S(x) =$ “y solving

$$\min_{y\in\mathbb{R}} \quad -xy$$
$$s.t \quad x^2(y^2-1) \leq 0 \text{ ”}$$

A trivial example

Lower level problem:

$$\begin{array}{cc} \min_{y\in\mathbb{R}} & -x.y \\ s.t & x^2(y^2-1)\leq 0 \text{ ''} \end{array}$$

Note that the solution map of this convex problem is

$$S(x) := \begin{cases} \{1\} & x<0 \\ \{-1\} & x>0 \\ \mathbb{R} & x=0 \end{cases}$$

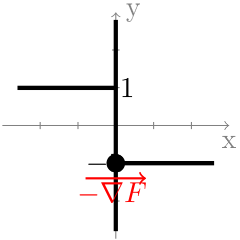
Thus for each $x \neq 0$ there is a unique associated solution of the lower level problem

A trivial example

Lower level problem:

$$\min_{y\in\mathbb{R}} \quad -xy$$
$$s.t \quad x^2(y^2-1)\leq 0 \text{ "}$$

Note that the solution map of this convex problem is



A trivial example

Consider the following simple bilevel problem

$$\begin{array}{cl} \text{“}\min_{x\in\mathbb{R}}\text{”} & -x.y \\ s.t. & \left\{\begin{array}{l} x \in [-1,1] \\ y \in S(x) \end{array}\right. \end{array}$$

with $S(x) =$ “y solving

$$S(x) := \begin{cases} \{1\} & x < 0 \\ \{-1\} & x > 0 \\ \mathbb{R} & x = 0 \end{cases}$$

Ambiguity: Optimistic approach

An *Optimistic Bilevel Problem* consists in an **upper-level/leader's problem**

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} \min_{y\in\mathbb{R}^m} & F(x,y) \\ s.t. & \begin{cases} x \in X \\ y \in S(x) \end{cases} \end{array}$$

where $\emptyset \neq X \subset \mathbb{R}^n$ and $S(x)$ stands for the solution set of its **lower-level/follower's problem**

$$\begin{array}{rl} \min_{y\in\mathbb{R}^m} & f(x,y) \\ s.t & g(x,y) \leq 0 \end{array}$$

Ambiguity: Pessimistic approach

An *Pessimistic Bilevel Problem* consists in an **upper-level/leader's problem**

$$\min_{x\in\mathbb{R}^n} \max_{y\in\mathbb{R}^m} \quad F(x,y) \qquad s.t. \quad \begin{cases} x \in X \\ y \in S(x) \end{cases}$$

where $\emptyset \neq X \subset \mathbb{R}^n$ and $S(x)$ stands for the solution set of its **lower-level/follower's problem**

$$\min_{y\in\mathbb{R}^m} \quad f(x,y) \qquad s.t \quad g(x,y) \leq 0$$

Ambiguity: the most simple

And of course the "confortable situation" corresponds to the case of a unique response

$$\forall\, x \in X, \quad S(x) = \{y(x)\}.$$

Then

$$\begin{array}{ll} \min_{x\in\mathbb{R}^n} & F(x, y(x)) \\ s.t. & \{\ x \in X \end{array}$$

Ambiguity: the most simple

And of course the "confortable situation" corresponds to the case of a unique response

$$\forall\, x \in X, \quad S(x) = \{y(x)\}.$$

Then

$$\begin{array}{cc} \min_{x\in\mathbb{R}^n} & F(x, y(x)) \\ s.t. & \left\{\ x \in X\right. \end{array}$$

For example when

for any x, $g(x,\cdot)$ is quasiconvex and $f(x,\cdot)$ is strictly convex.

An *"Selection-type" Bilevel Problem* consists in an **upper-level/leader's problem**

$$\min_{x\in\mathbb{R}^n} \quad F(x, y(x))$$
$$s.t. \quad \begin{cases} x \in X \\ y(x) \textit{ is a uniquely determined selection of } S(x) \end{cases}$$

J. Escobar & A. Jofré, *Equilibrium Analysis of Electricity Auctions* (2011)

Ambiguity: The new probabilistic approach

In one of the Elevator pitches (Monday), D.Salas and A. Svensson proposed a **probabilistic approach**:

- *Consider a probability on the different possible follower's reactions*
- *Minimize the expectation of the leader(s)*

An alternative point of view

Instead of considering the previous (optimistic) formulation of BL:

$$\min_{x\in\mathbb{R}^n} \min_{y\in\mathbb{R}^m} \quad F(x,y) \qquad s.t. \quad \begin{cases} x \in X \\ y \in S(x) \end{cases}$$

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$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} \min_{y\in\mathbb{R}^m} & F(x,y) \\ s.t. & \left\{ \begin{array}{l} x \in X \\ y \in S(x) \end{array} \right. \end{array}$$

one can define the (optimistic) value function

$$\varphi_{min}(x) = \min_y \{F(x,y) : g(x,y) \leq 0\} \tag{1}$$

and the Bl problem becomes

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} & \varphi_{min}(x) \\ s.t. & x \in X \end{array}$$

An alternative point of view

Instead of considering the previous (pessimistic) formulation of BL:

$$\min_{x\in\mathbb{R}^n}\max_{y\in\mathbb{R}^m} \quad F(x,y) \qquad s.t. \quad \begin{cases} x\in X \\ y\in S(x) \end{cases}$$

An alternative point of view

Instead of considering the previous (pessimistic) formulation of BL:

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} \max_{y\in\mathbb{R}^m} & F(x,y) \\ s.t. & \begin{cases} x \in X \\ y \in S(x) \end{cases} \end{array}$$

one can define the (pessimistic) value function

$$\varphi_{max}(x) = \max_{y}\{F(x,y) : g(x,y) \leq 0\} \tag{2}$$

and the Bl problem becomes

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} & \varphi_{max}(x) \\ s.t. & x \in X \end{array}$$

An alternative point of view

This is the point of view presented in Stephan Dempe's book:

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} \min/\max_{y\in\mathbb{R}^m} & F(x,y) \\ s.t. & \left\{\begin{array}{l} x\in X \\ y\in S(x) \end{array}\right. \end{array} \quad \textbf{vs} \quad \begin{array}{rl} \min_{x\in\mathbb{R}^n} & \varphi_{min/max}(x) \\ s.t. & x\in X \end{array}$$

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It immediately raises the question

What is a solution??

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This is the point of view presented in Stephan Dempe's book:

$$\begin{array}{ll} \min_{x\in\mathbb{R}^n} \min / \max_{y\in\mathbb{R}^m} & F(x,y) \\ s.t. & \begin{cases} x \in X \\ y \in S(x) \end{cases} \end{array} \quad \textbf{vs} \quad \begin{array}{ll} \min_{x\in\mathbb{R}^n} & \varphi_{min/max}(x) \\ s.t. & x \in X \end{array}$$

It immediately raises the question

What is a solution??

- *an optimal x = leader's optimal strategy?*
- *an optimal couple (x,y) = couple of strategies of leader and follower?*

Real life...

Actually usually when considering BL

$$\min_{x\in\mathbb{R}^n} \min_{y\in\mathbb{R}^m} \quad F(x,y)$$
$$s.t. \quad \begin{cases} x \in X \\ y \in S(x) \end{cases}$$

people say

- *Step A: the leader plays first*
- *Step B: the follower reacts*

But in real life it's a little bit more complex....

Real life...

Actually in real life, when considering BL

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} \min_{y\in\mathbb{R}^m} & F(x,y) \\ s.t. & \left\{ \begin{array}{l} x \in X \\ y \in S(x) \end{array} \right. \end{array}$$

We only work for the leader!!

Real life...

Actually in real life, when considering BL

$$\begin{array}{rl} \min_{x\in\mathbb{R}^n} \min_{y\in\mathbb{R}^m} & F(x,y) \\ s.t. & \left\{ \begin{array}{l} x \in X \\ y \in S(x) \end{array} \right. \end{array}$$

We only work for the leader!! Indeed

- *the leader has a model of the follower's reaction: optimistic or pessimistic and*
- *Step 1: we compute a solution x or (x,y) of the BL model*
- *Step 2: the leader plays x*
- *Step 3: the follower decides to play...whatever he wants!!!*

Definition

The Mangasarian-Fromovitz constraint qualification (MFCQ) is satisfied at $(\overline{x}, \overline{y})$ with $\overline{y}$ feasible point of the problem

$$\min_y \{f(x,y) : g(x,y) \leq 0\}$$

if the system

$$\nabla_y g_i(\overline{x}, \overline{y}) d < 0 \ \forall \ i \in I(\overline{x}, \overline{y}) := \{j : g_j(\overline{x}, \overline{y}) = 0\}$$

has a solution.

An existence result (cont.)

Assume that $X = \{x \in \mathbb{R}^n \ : \ G(x) \leq 0\}$

Theorem (Bank, Guddat, Klatte, Kummer, Tammer (83))

Let $\overline{x}$ with $G(\overline{x}) \leq 0$ be fixed.

- *the set $\{(x,y) : g(x,y) \leq 0\}$ is not empty and compact;*
- *at each point $(\overline{x}, \overline{y}) \in \mathbf{gph} S$ with $G(\overline{x}) \leq 0$, assumption (MFCQ) is satisfied;*

then, the set-valued map $S(\cdot)$ is upper semicontinuous at $(\overline{x}, \overline{y})$ and the function $\varphi_o(\cdot)$ is continuous at $\overline{x}$.

An existence result (cont.)

Theorem

Assume that

- *the set* $\{(x,y) : g(x,y) \leq 0\}$ *is not empty and compact;*
- *at each point* $(\overline{x}, \overline{y}) \in \mathbf{gph} S$ *with* $G(\overline{x}) \leq 0$, *assumptions (MFCQ) is satisfied;*
- *the set* $\{x : G(x) \leq 0\}$ *is not empty and compact,*

then optimistic bilevel problem has a (global) optimal solution.

Bilevel problems and MPCC reformulation

We consider a Bilevel Problem consisting in an **upper-level / leader's problem**

$$\text{“}\min_{x\in\mathbb{R}^n}\text{”}\, F(x,y)$$
$$s.t.\;\; y \in S(x),\, x \in X$$

where $\emptyset \neq X \subset \mathbb{R}^n$, and $S(x)$ stands for the solution of its **lower-level / follower's problem**

$$\min_{y\in\mathbb{R}^m} \, f(x,y)$$
$$s.t\; g(x,y) \leq 0$$

which we assume to be convex and smooth, i.e. $\forall x \in X$, the functions $f(x,\cdot)$ and $g_i(x,\cdot)$ are smooth convex functions, and the gradients $\nabla_y g_i, \nabla_y f$ are continuous, $i = 1, ..., p$.

MPCC reformulation

Replacing the lower-level problem by its KKT conditions, gives place to a Mathematical Program with Complementarity Constraints.

Bilevel

$$\text{“}\min_{x \in X}\text{”} F(x,y)$$
$$s.t. \;\; y \in S(x)$$

with $S(x) =$ “y solving

$$\min_{y \in \mathbb{R}^m} \; f(x,y)$$
$$s.t \;\; g(x,y) \leq 0 \;\; \text{”}$$

MPCC

$$\text{“}\min_{x \in X}\text{”} F(x,y)$$
$$s.t. \;\; (y,u) \in KKT(x)$$

with $KKT(x) =$ “(y,u) solving

$$\begin{cases} \nabla_y f(x,y) + u^T \nabla_y g(x,y) = 0 \\ 0 \leq u \perp -g(x,y) \geq 0 \;\; \text{”} \end{cases}$$

We write $\Lambda(x,y)$ for the set of u satisfying $(y,u) \in KKT(x)$.

Example 1

Consider the following Bilevel problem and its MPCC reformulation

Bilevel

$$\text{“}\min_{x\in[-1,1]}\text{”}\, x \qquad s.t.\ \ y \in S(x)$$

with $S(x) =$ “y solving

$$\min_{y\in\mathbb{R}} \quad xy \qquad s.t\ x^2(y^2-1) \le 0 \ \text{”}$$

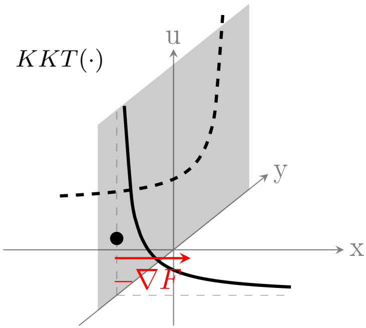
MPCC

$$\text{“}\min_{x\in[-1,1]}\text{”}\, x \qquad s.t.\ \ (y,u) \in KKT(x)$$

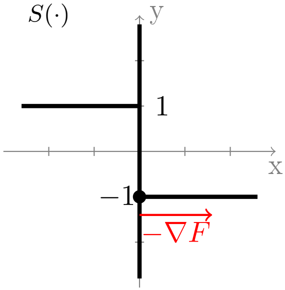
with $KKT(x) =$ “(y,u) solving

$$\begin{cases} x + u \cdot 2yx^2 = 0 \\ 0 \le u \perp -x^2(y^2-1) \ge 0 \ \text{”} \end{cases}$$

1. $(0,-1,u)$ is a local solution of “MPCC”, for any $u \in \Lambda(0,-1) = \mathbb{R}_+$
2. $(0,-1)$ is NOT a local solution of “Bilevel”



(a) $(0,-1,u)$ is a local solution of MPCC, $\forall u \in \mathbb{R}_+$.



(b) $(0,-1)$ isn't a local solution of the Bilevel problem.

Optimistic and Pessimistic approaches

The optimistic Bilevel (OB) is

$$\min_x \min_y F(x,y)$$
$$s.t.\ \ y \in S(x), x \in X.$$

The pessimistic Bilevel (PB) is

$$\min_x \max_y F(x,y)$$
$$s.t.\ \ y \in S(x), x \in X.$$

Optimistic and Pessimistic approaches

The optimistic Bilevel (OB) is

$$\min_x \min_y F(x,y) \quad s.t. \; y \in S(x), x \in X.$$

The optimistic MPCC (OMPCC):

$$\min_x \min_y F(x,y) \quad s.t. \; (y,u) \in KKT(x), x \in X.$$

The pessimistic Bilevel (PB) is

$$\min_x \max_y F(x,y) \quad s.t. \; y \in S(x), x \in X.$$

The pessimistic MPCC (PMPCC):

$$\min_x \max_y F(x,y) \quad s.t. \; (y,u) \in KKT(x), x \in X.$$

Optimistic approach

Is bilevel programming a special case of a MPCC?

S. Dempe -J. Dutta (2012 Math. Prog.)

$$\min_x \min_y F(x,y)$$
$$\text{s.t. } y \in S(x), x \in X.$$

Local solutions for in optimistic approach

Definition

A **local (resp. global) solution** of (OB) is a point $(\bar{x}, \bar{y}) \in Gr(S)$ if there exists $U \in \mathcal{N}(\bar{x}, \bar{y})$ (resp. $U = \mathbb{R}^n \times \mathbb{R}^m$) such that

$$F(\bar{x}, \bar{y}) \leq F(x, y), \quad \forall (x, y) \in U \cap Gr(S).$$

Definition

A **local (resp. global) solution** for (OMPCC) is a triplet $(\bar{x}, \bar{y}, \bar{u}) \in Gr(KKT)$ such that there exists $U \in \mathcal{N}(\bar{x}, \bar{y}, \bar{u})$ (resp. $U = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$) with

$$F(\bar{x}, \bar{y}) \leq F(x, y), \quad \forall (x, y, u) \in U \cap Gr(KKT).$$

In Dempe-Dutta it was considered the Slater type constraint qualification for a parameter $x \in X$:

$$\textbf{Slater: } \exists y(x) \in \mathbb{R}^m \text{ s.t. } g_i(x, y(x)) < 0,\ \forall i = 1, .., p.$$

Theorem 1 Dempe-Dutta (2012)

Assume the convexity condition and Slater's CQ at $\bar{x}$.

1. If $(\bar{x}, \bar{y})$ is a local solution for (OB), then for each $\bar{u} \in \Lambda(\bar{x}, \bar{y})$, $(\bar{x}, \bar{y}, \bar{u})$ is a local solution for (OMPCC).
2. Conversely, assume that Slater's CQ holds on a neighbourhood of $\bar{x}$, $\Lambda(\bar{x}, \bar{y}) \neq \emptyset$, and $(\bar{x}, \bar{y}, u)$ is a local solution of (OMPCC) for every $u \in \Lambda(\bar{x}, \bar{y})$. Then $(\bar{x}, \bar{y})$ is a local solution of (OB).

Under the convexity assumption and some CQ ensuring $KKT(x) \neq \emptyset$, $\forall x \in X$:

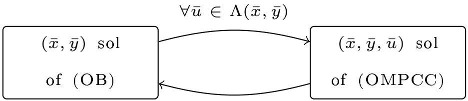


Figure: Global solution comparison in optimistic approach

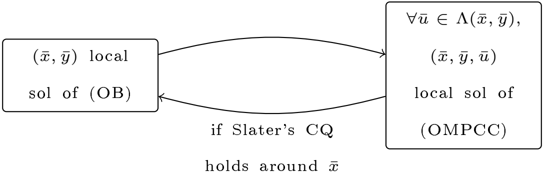


Figure: Local solution comparison in optimistic approach

Example 1 (optimistic)

Consider the following optimistic Bilevel problem

$$\min_{x\in[-1,1]} \min_{y} x$$
$$\text{s.t. } y \in S(x),\ x \in \mathbb{R}$$

with lower-level

$$\min_{y} \ -xy$$
$$\text{s.t } x^2(y^2-1) \leq 0.$$

1. $(0,-1,u)$ is a local solution of (OMPCC), for any $u \in \Lambda(0,-1) = \mathbb{R}_+$
2. $(0,-1)$ is NOT a local solution of (OB).

Pessimistic Approach

Is bilevel programming a special case of a (MPCC)?

Aussel - Svensson (2019 - J. Optim. Theory Appl.)

$$\min_x \max_y F(x,y)$$
$$\text{s.t. } y \in S(x), x \in X.$$

Definition

A pair $(\bar{x}, \bar{y})$ is said to be a *local (resp. global) solution* for (PB), if $(\bar{x}, \bar{y}) \in Gr(S_p)$ and $\exists U \in \mathcal{N}(\bar{x}, \bar{y})$ such that

$$F(\bar{x}, \bar{y}) \leq F(x, y), \quad \forall (x, y) \in U \cap Gr(S_p). \tag{3}$$

where $S_p(x) := argmax_y \{F(x, y) \mid y \in S(x)\}$.

Definition

A triplet $(\bar{x}, \bar{y}, \bar{u})$ is said to be a *local (resp. global) solution* for (PMPCC), if $(\bar{x}, \bar{y}, \bar{u}) \in Gr(KKT_p)$ and $\exists U \in \mathcal{N}(\bar{x}, \bar{y}, \bar{u})$ such that

$$F(\bar{x}, \bar{y}) \leq F(x, y), \quad \forall (x, y, u) \in U \cap Gr(KKT_p). \tag{4}$$

where $KKT_p(x) := argmax_{y,u} \{F(x, y) \mid (y, u) \in KKT(x)\}$.

Results for the pessimistic case

Theorem 2

Assume the convexity condition and that $KKT(x) \neq \emptyset, \forall x \in X$.

1. If $(\bar{x}, \bar{y})$ is a local solution for (PB), then for each $\bar{u} \in \Lambda(\bar{x}, \bar{y})$, $(\bar{x}, \bar{y}, \bar{u})$ is a local solution for (PMPCC).
2. Conversely, assume that one of the following condition are satisfied:
    1. The multifunction KKT_p is LSC around $(\bar{x}, \bar{y}, \bar{u})$ and $(\bar{x}, \bar{y}, \bar{u})$ is a local solution of (PB).
    2. Slater's CQ holds on a neighbourhood of $\bar{x}$, $\Lambda(\bar{x}, \bar{y}) \neq \emptyset$, and for every $u \in \Lambda(\bar{x}, \bar{y})$, $(\bar{x}, \bar{y}, u)$ is a local solution of (PMPCC).

    Then $(\bar{x}, \bar{y})$ is a local solution of (PB).

Example 1 (pessimistic)

Consider the following pessimistic Bilevel problem

$$\min_{x\in[-1,1]} \max_{y} x$$
$$\text{s.t. } y \in S(x),\ x \in \mathbb{R}$$

with lower-level

$$\min_{y} \ -xy$$
$$\text{s.t } x^2(y^2-1) \leq 0.$$

1. $(0,-1,u)$ is a local solution of (PMPCC), for any $u \in \Lambda(0,-1) = \mathbb{R}_+$
2. $(0,-1)$ is NOT a local solution of (PB).

Example 2

Consider the following Bilevel problem

$$\text{“}\min_x\text{”} x$$
$$s.t. \quad y \in S(x)$$

with $S(x)$ the solution of the lower-level problem

$$\min_y \{-y \mid x + y \le 0, y \le 0\}$$

Even though Slater's CQ holds, we have

1. $(0,0,u_1,u_2)$ with $(u_1,u_2) \in \Lambda(0,0) = \{(\lambda, 1-\lambda) \mid \lambda \in [0,1]\}$ is a local solution of "(MPCC)", iff $u_1 \neq 0$,
2. $(0,0)$ is NOT a local solution for "(B)".

Under the convexity assumption and some (CQ) ensuring $KKT(x) \neq \emptyset$, $\forall x \in X$:

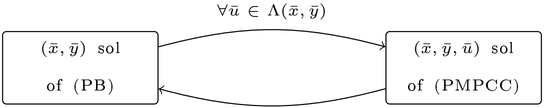


Figure: Global solution comparison in pessimistic approach

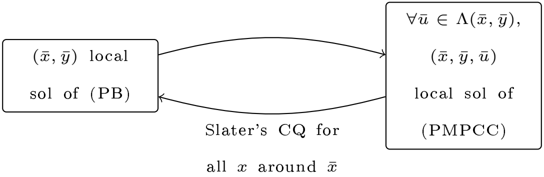


Figure: Local solutions comparison in pessimistic approach

An introduction to MLFG

- *A Nash equilibrium problem is a* **noncooperative game** *in which the* **decision function** *(cost/benefit) of each player depends on the decision of the other players.*

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Denote by N the number of players and each player i controls variables $x^i \in \mathbb{R}^{n_i}$. The "total strategy vector" is x which will be often denoted by

$$x = (x^i, x^{-i}).$$

where x^{-i} is the strategy vector of the *other players*.

- The strategy of player i belongs to a strategy set

$$x^i \in X_i$$

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- Given the strategies x^{-i} of the other players, the aim of player i is to choose a strategy x^i solving

$$P_i(x^{-i}) \qquad \begin{array}{ll} \max & \theta_i(x^i, x^{-i}) \\ s.t. & x^i \in X_i \end{array}$$

where $\theta_i(\cdot, x^{-i}) : \mathbb{R}^{n_i} \to \mathbb{R}$ is the decision function for player i.

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where $\theta_i(\cdot, x^{-i}) : \mathbb{R}^{n_i} \to \mathbb{R}$ is the decision function for player i.

- A vector $\bar{x}$ is a *Nash Equilibrium* if

$$\textit{for any } i, \qquad \bar{x}^i \quad \textit{solves} \quad P_i(\bar{x}^{-i}).$$

- *A generalized Nash equilibrium problem (GNEP) is a* **noncooperative game** *in which the* **decision function and strategy set** *of each player depend on the decision of the other players.*

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- A vector $\bar{x}$ is a Generalized Nash Equilibrium if

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Generalized Nash game (GNEP):

$$\begin{array}{|c|}\hline \boxed{\begin{array}{ll} \min\limits_{x_1} & \theta_1(x) \\ \text{s.t.} & \{\ x_1 \in X_1(x_{-1}) \end{array}} \quad \cdots \quad \boxed{\begin{array}{ll} \min\limits_{x_n} & \theta_n(x) \\ \text{s.t.} & \{\ x_n \in X_n(x_{-n}) \end{array}} \\ \hline \end{array}$$

A classical existence result

Theorem (Ichiishi-Quinzii 1983)

Let a GNEP be given and suppose that

1. *For each $\nu = 1, ..., N$ there exist a nonempty, convex and compact set $K_\nu \subset \mathbb{R}^{n_\nu}$ such that the point-to-set map $X^\nu : K_{-\nu} \rightrightarrows K_\nu$, is both upper and lower semicontinuous with nonempty closed and convex values, where $K_{-\nu} := \prod_{\nu' \neq \nu} K_\nu$.*
2. *For every player ν, the function θ^ν is continuous and $\theta^\nu(\cdot, x^{-\nu})$ is quasi-convex on $X^\nu(x^{-\nu})$.*

Then a generalized Nash equilibrium exists.

Note that in Aussel-Dutta (2008) an alternative proof of existence of equilibria has been given, under the assumption of the Rosen's law, by using the normal approach technique.

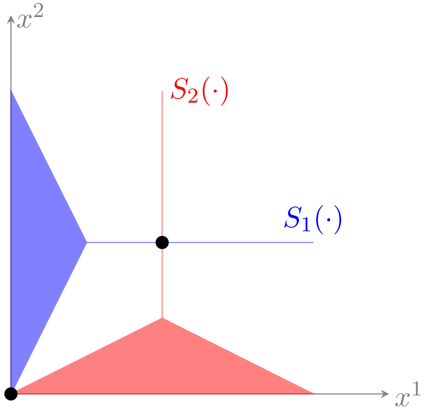
Structure of the set of GNEPs

Example

Let $x = (x^1, x^2) \in [0,4]^2$ and $f^\nu(x) := d_{T_\nu}(x)^2$, where T_1 is the triangle with vertices $(0,0)$, $(0,4)$ and $(1,2)$, and T_2 is the triangle whose vertices are $(0,0)$, $(4,0)$ and $(2,1)$. Let $S_\nu(x^{-\nu}) := \text{argmin}_{x^\nu} \left\{ f^\nu(x^1,x^2) \mid x^\nu \in [0,4] \right\}$. We see that

- $S_1(x^2) = \left\{ x^1 \in [0,4] \mid (x^1,x^2) \in T_1 \right\}$ for $x^2 \in [0,1]$
- $S_1(x^2) = \{2\}$ for all $x^2 \in (1,4])$
- $S_2(x^1) = \left\{ x^2 \in [0,4] \mid (x^1,x^2) \in T_2 \right\}$ for $x^1 \in [0,1]$
- $S_2(x^1) = \{2\}$ for all $x^1 \in (1,4])$.

Structure of the set of GNEPs (cont.)



Multi-Leader-Follower-Game (MLFG):

$$\min_{\substack{x_1 \\ y_1,..,y_p}} \quad \theta_1(x,y) \quad \text{s.t.} \quad \begin{cases} x_1 \in X_1(x_{-1}) \\ y \in Y(x) \end{cases} \qquad \cdots \qquad \min_{\substack{x_n \\ y_1,..,y_p}} \quad \theta_n(x,y) \quad \text{s.t.} \quad \begin{cases} x_n \in X_n(x_{-n}) \\ y \in Y(x) \end{cases}$$

$$\downarrow\uparrow \qquad\qquad \downarrow\uparrow$$

$$\min_{y_1,..,y_p} \quad \phi_1(x,y) \quad \text{s.t.} \quad \{\ y \in Y(x) \qquad \cdots \qquad \min_{y_1,..,y_p} \quad \phi_p(x,y) \quad \text{s.t.} \quad \{\ y \in Y(x)$$

Single-Leader-Multi-Follower-Game (SLMFG):

$$\min_{\substack{x \\ y_1,..,y_p}} \quad \theta_1(x,y) \qquad \text{s.t.} \quad \begin{cases} x \in X \\ y \in Y(x) \end{cases}$$

$$\downarrow\uparrow$$

$$\boxed{\min_{y_1,..,y_p} \ \phi_1(x,y) \ \text{ s.t. } \{\ y \in Y(x)} \quad \ldots \quad \boxed{\min_{y_1,..,y_p} \ \phi_p(x,y) \ \text{ s.t. } \{\ y \in Y(x)}$$

Multi-Leader-Single-Follower-Game (MLSFG):

$$\begin{array}{ll} \min\limits_{\substack{x_1 \\ y_1,..,y_p}} & \theta_1(x,y) \\ \text{s.t.} & \left\{ \begin{array}{l} x_1 \in X_1(x_{-1}) \\ y \in Y(x) \end{array} \right. \end{array} \quad \cdots \quad \begin{array}{ll} \min\limits_{\substack{x_n \\ y_1,..,y_p}} & \theta_n(x,y) \\ \text{s.t.} & \left\{ \begin{array}{l} x_n \in X_n(x_{-n}) \\ y \in Y(x) \end{array} \right. \end{array}$$

$$\downarrow\uparrow$$

$$\begin{array}{ll} \min_{y_1,..,y_p} & \phi_1(x,y) \\ \text{s.t.} & \{ \; y \in Y(x) \end{array}$$

MLSF game: ill-posedness

$$\min_{x_1,y} \quad \theta_1(x_1,x_2,y) = x_1.y \qquad\qquad \min_{x_2,y} \quad \theta_1(x_1,x_2,y) = -x_2.y$$

$$s.t. \quad \begin{cases} x_1 \in [0,1] \\ y \in S(x_1,x_2) \end{cases} \qquad\qquad s.t. \quad \begin{cases} x_2 \in [0,1] \\ y \in S(x_1,x_2) \end{cases}$$

with

$$\min_y \quad f(x_1,x_2,y) = \frac{1}{3}y^3 - (x_1+x_2)^2 y$$

$$s.t. \quad y \in \mathbb{R}$$

MLSF game: ill-posedness

$$\min_{x_1,y} \quad \theta_1(x_1,x_2,y) = x_1.y \qquad s.t. \quad \begin{cases} x_1 \in [0,1] \\ y \in S(x_1,x_2) \end{cases}$$

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Exercise: Please analyse this small example...

MLSF game: ill-posedness

The follower problem first

$$\begin{array}{rl} \min_y & f(x_1, x_2, y) = \frac{1}{3}y^3 - (x_1 + x_2)^2 y \\ s.t. & y \in \mathbb{R} \end{array}$$

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The solution map of this follower problem is

$$S(x_1, x_2) = \{y_1 = x_1 + x_2, y_2 = -x_1 - x_2\}.$$

MLSF game: ill-posedness

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The leader 1 problem

$$\theta_1(x,y) = x_1.y = \begin{cases} x_1^2 + x_1.x_2 & \textit{if } y = y_1 \\ -x_1^2 - x_1.x_2 & \textit{if } y = y_2 \end{cases}$$

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Thus the response function of player 1 is

$$\mathbb{R}_1(x_2) = \begin{cases} \{0\} & \text{if } y = y_1 \text{ with a payoff} = 0 \\ \{1\} & \text{if } y = y_2 \text{ with a payoff} = -1 - x_2 \end{cases}$$

MLSF game: ill-posedness

The solution map of this follower problem is

$$S(x_1, x_2) = \{y_1 = x_1 + x_2, y_2 = -x_1 - x_2\}.$$

The leader 2 problem

$$\theta_1(x, y) = -x_2.y = \begin{cases} -x_1^2 - x_1.x_2 & \textit{if } y = y_1 \\ x_1^2 + x_1.x_2 & \textit{if } y = y_2 \end{cases}$$

MLSF game: ill-posedness

The solution map of this follower problem is

$$S(x_1, x_2) = \{y_1 = x_1 + x_2, y_2 = -x_1 - x_2\}.$$

The leader 2 problem

$$\theta_1(x, y) = -x_2.y = \begin{cases} -x_1^2 - x_1.x_2 & \textit{if } y = y_1 \\ x_1^2 + x_1.x_2 & \textit{if } y = y_2 \end{cases}$$

Thus the response function of player 1 is

$$\mathbb{R}_2(x_1) = \begin{cases} \{1\} & \textit{if } y = y_1 \textit{ with a payoff} = -1 - x_1 \\ \{0\} & \textit{if } y = y_2 \textit{ with a payoff} = 0 \end{cases}$$

MLSF game: ill-posedness

$$\mathbb{R}_1(x_2) = \begin{cases} \{(0, y = y_1)\} & \textit{with a payoff} = 0 \\ \{(1, y = y_2)\} & \textit{with a payoff} = -1 - x_2 \end{cases}$$

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MLSF game: ill-posedness

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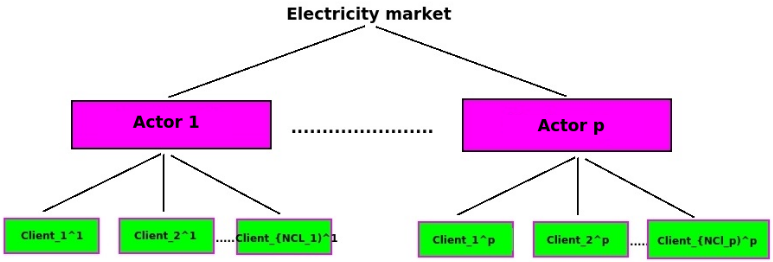
So the Nash equilibrium will be $(x_1, x_2) = (1, 1)$ but....

A final model

For the Demand-side management, we recently introduced the Multi-Leader-Disjoint-Follower game



Just one example [Pang-Fukushima 05]

Let us consider a 2-leader-single-follower game:

$$\min_{x_1,y} \quad \tfrac{1}{2}x_1 + y \qquad \begin{cases} x_1 \in [0,1] \\ y \in S(x_1,x_2) \end{cases}$$

$$\min_{x_2,y} \quad -\tfrac{1}{2}x_2 - y \qquad \begin{cases} x_2 \in [0,1] \\ y \in S(x_1,x_2) \end{cases}$$

where $S(x_1,x_2)$ is the solution map of

$$\min_{y\geq 0} \quad y(-1+x_1+x_2) + \frac{1}{2}y^2$$

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$$min_{x_2, y_2} \quad -\frac{1}{2}x_2 - y_2 \qquad \left\{ \begin{array}{l} x_2 \in [0,1] \\ y_2 \in S(x_1, x_2) \end{array} \right.$$

where $S(x_1, x_2)$ is the solution map of

$$min_{y \geq 0} \quad y(-1 + x_1 + x_2) + \frac{1}{2}y^2$$

Actually $S(x_1, x_2) = \max\{0, 1 - x_1 - x_2\}$ thus the problem becomes

$$min_{x_1,y_1} \quad \frac{1}{2}x_1 + y_1 \qquad \left\{ \begin{array}{l} x_1 \in [0,1] \\ y_1 = \max\{0, 1 - x_1 - x_2\} \end{array} \right.$$

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Then the Response maps are

$$\mathcal{R}_1(x_2) = \{1 - x_2\} \quad \text{and} \quad \mathcal{R}_2(x_1) = \left\{ \begin{array}{ll} \{0\} & x_1 \in [0, \frac{1}{2}[\\ \{0,1\} & x_1 = \frac{1}{2} \\ \{1\} & x_1 \in]\frac{1}{2}, 1] \end{array} \right.$$

and thus there is no Nash equilibrium.......

But let us consider the slightly modified problem.......

$$min_{x_1,y_1} \quad \frac{1}{2}x_1 + y_1 \qquad\qquad min_{x_2,y_2} \quad -\frac{1}{2}x_2 - y_2$$

$$\left\{\begin{array}{l} x_1 \in [0,1] \\ y_1 = \max\{0, 1 - x_1 - x_2\} \\ y_2 = \max\{0, 1 - x_1 - x_2\} \end{array}\right. \qquad\qquad \left\{\begin{array}{l} x_2 \in [0,1] \\ y_1 = \max\{0, 1 - x_1 - x_2\} \\ y_2 = \max\{0, 1 - x_1 - x_2\} \end{array}\right.$$

But let us consider the slightly modified problem.......

$$min_{x_1,y_1} \quad \frac{1}{2}x_1 + y_1 \qquad \left\{ \begin{array}{l} x_1 \in [0,1] \\ y_1 = \max\{0, 1 - x_1 - x_2\} \\ y_2 = \max\{0, 1 - x_1 - x_2\} \end{array} \right.$$

$$min_{x_2,y_2} \quad -\frac{1}{2}x_2 - y_2 \qquad \left\{ \begin{array}{l} x_2 \in [0,1] \\ y_1 = \max\{0, 1 - x_1 - x_2\} \\ y_2 = \max\{0, 1 - x_1 - x_2\} \end{array} \right.$$

that can be proved to have a (unique) Nash equilibrium namely $(x_1, x_2) = (0, 1)$ with $y_1 = y_2 = 0$!!!!

The kind of "trick" is called "All Equilibrium approach" and has been introduced in A.A. Kulkarni & U.V. Shanbhag, *A Shared-Constraint Approach to Multi-Leader Multi-Follower Games*, Set-Valued Var. Anal (2014).

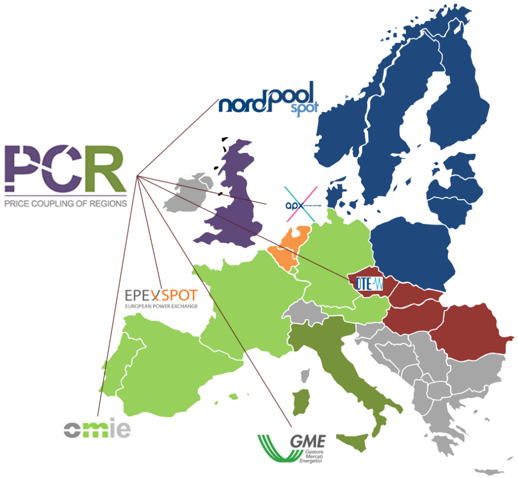
They proved that every Nash equilibirum (initial problem) is a Nash equilibrium for the "all equilibrium" formulation.

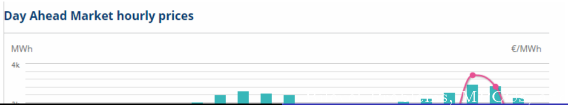
It corresponds to the case where each leader takes into account the conjectures regarding the follower decision made by all other leaders....

Some motivation examples

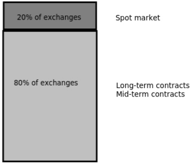
Electricity markets

A short introduction to electricity markets





Volume of exchanges



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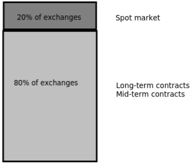
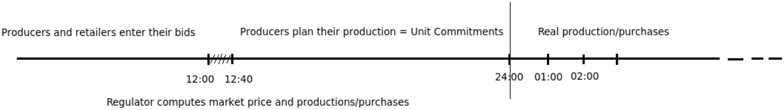


Bid schedule of the spot market



- electricity market consists of
  i) generators/consumers $i \in \mathcal{N}$ respect their own interests in competition with others
  ii) market operator (ISO) who maintain energy generation and load balance, and protect public welfare

- the ISO has to consider:
  ii) quantities q_i of generated/consumed electricity
  iii) electricity dispatch t_e with respect to transmission capacities

Modeling an Electricity Markets

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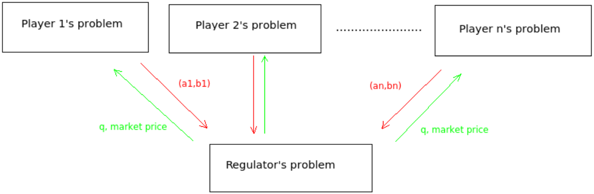
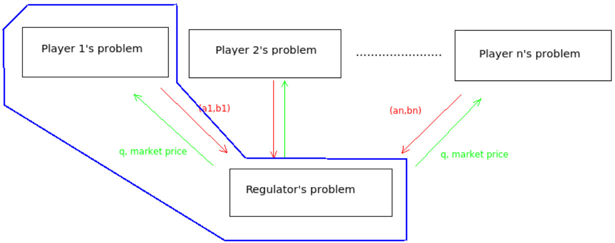
  iii) electricity dispatch t_e with respect to transmission capacities

- since 1990s, Generalized Nash equilibrium problem is the most popular way of modeling spot electricity markets or, more precisely, Multi-leader-common-follower game

Multi-Leader-Common-Follower game



A classical problem (of a producer) is the best response search



Models with linear unit bid functions

- **Electricity markets without transmission losses:**
  X. Hu & D. Ralph, Using EPECs to Model Bilevel Games in Restructured Electricity Markets with Locational Prices, *Operations Research (2007). bid-on-a-only*

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- **Electricity markets with transmission losses:**
  - *Henrion, R., Outrata, J. & Surowiec, T.,* Analysis of M-stationary points to an EPEC modeling oligopolistic competition in an electricity spot market, *ESAIM: COCV (2012). M-stationary points*
  - *D. A., R. Correa & M. Marechal* Spot electricity market with transmission losses, *J. Industrial Manag. Optim (2013). existence of Nash equil., case of a two island model*
  - *D.A., M. Cervinka & M. Marechal,* Deregulated electricity markets with thermal losses and production bounds, *RAIRO (2016) production bounds, well-posedness of model*

Some references on the topic (cont.)

- **Best response in electricity markets:**
  - *E. Anderson and A. Philpott*, Optimal Offer Construction in Electricity Markets, *Mathematics of Operations Research (2002). Linear bid function - necessary optimality cond. for local best response in time dependent case*
  - *D. Aussel, P. Bendotti and M. Pištěk, Nash Equilibrium in Pay-as-bid Electricity Market : Part 2 - Best Response of Producer, Optimization (2017) linear unit bid function, explicit formula for best response*

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  - *D. Aussel, P. Bendotti and M. Pištěk, Nash Equilibrium in Pay-as-bid Electricity Market : Part 2 - Best Response of Producer, Optimization (2017) linear unit bid function, explicit formula for best response*

- **Explicit formula for equilibria**
  D. Aussel, P. Bendotti and M. Pištěk, Nash Equilibrium in Pay-as-bid Electricity Market : Part 1 - Existence and Characterisation, Optimization (2017) explicit formula for equilibria

But also...

- **Non a priori structured bid functions**
  - *Escobar, J.F. and Jofré, A., Monopolistic competition in electricity networks with resistance losses, Econom. Theory 44 (2010).*
  - *Escobar, J.F. and Jofré, A., Equilibrium analysis of electricity auctions, preprint (2014).*
  - *E. Anderson, P. Holmberg and A. Philpott, Mixed strategies in discriminatory divisible-good auctions, The RAND Journal of Economics (2013). necessary optimality cond. for local best response*

Notations

D. Aussel, P. Bendotti and M. Pištěk, Nash Equilibrium in Pay-as-bid Electricity Market : Optimization

Part 1 - Existence and Characterisation, 66:6 (2017)

Part 2 - Best Response of Producer, 66:6 (2017).

Let consider a fixed time instant and denote

- $D > 0$ be the overall energy demand of all consumers
- $\mathcal{N}$ be the set of producers
- $q_i \geq 0$ be the production of i-th producer, $i \in \mathcal{N}$

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We assume that producer $i \in \mathcal{N}$ provides to the ISO a quadratic bid function $a_i q_i + b_i q_i^2$ given by $a_i, b_i \geq 0$.

Similarly, let $A_i q_i + B_i q_i^2$ be the true production cost of i-th producer with $A_i \geq 0$ and $B_i > 0$ reflecting the increasing marginal cost of production.

Multi-Leader-Common-Follower game

Peculiarity of electricity markets is their bi-level structure:

$$P_i(a_{-i}, b_{-i}, D) \qquad \max_{a_i, b_i} \max_{q_i} \quad a_i q_i + b_i q_i^2 - (A_i q_i + B_i q_i^2)$$
$$\text{such that} \quad \begin{cases} a_i, b_i \geq 0 \\ (q_j)_{j \in \mathcal{N}} \in Q(a,b) \end{cases}$$

where set-valued mapping $Q(a,b)$ denotes solution set of

$$ISO(a,b,D) \qquad Q(a,b) = \underset{q}{\mathit{argmin}} \quad \textstyle\sum_{i \in \mathcal{N}} (a_i q_i + b_i q_i^2)$$
$$\text{such that} \quad \begin{cases} q_i \geq 0 \ , \ \forall i \in \mathcal{N} \\ \displaystyle\sum_{i \in \mathcal{N}} q_i = D \end{cases}$$

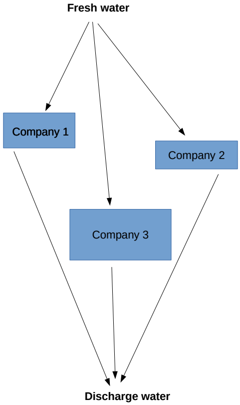
Some motivation examples

Industrial Eco-Parks

Example of water management

- In a geographical area, there are different companies $1, \ldots, n$
- Each of them is buying fresh water (high price) for their production processes
- Each company generates some "dirty water" and have to pay for discharge

Stand alone situation



How dos it work ?

The aims in designing Industrial Eco-park (IEP) are

a) Reduce cost of production of each company
b) Reduce the environmental impact of the whole production

Thus "Eco" of IEP is at the same time Economical and ecological

Example of water management

How to reach these aims?

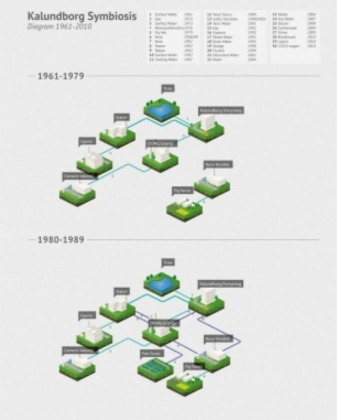
a) create a network (water tubes) between the companies

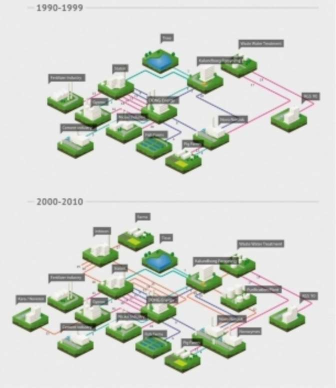
b) Eventually install some regeneration unit (cleaning of the water)

It is important to understand that this approach is not limited to water. It can be applied to vapor, gas, coaling fluids, human resources...

Kalundborg (Danemark)

An symbolic example of Industrial eco-park is Kalundborg (Danemark)





What is an « Eco-park » ?

In order to convince companies to participate to the Ecopark, our model should guarantee that:

a) **each company** will have a lower cost of production in Eco-park organization than in stand-alone organization

b) the eco-park organization must generate a **lower freshwater consumption** than with a stand-alone organization

MOO classical treatment

The Eco-park design was done through Multi-objective Optimization by the evaluation of Pareto fronts (Gold programming algorithms, scalarization...).

$$
\min \quad \begin{cases} \textit{Fresh water consumption} \\ \textit{Individual costs of producer } 1 \\ \vdots \\ \textit{Individual costs of producer } n \end{cases}
$$

$$
s.t. \quad \begin{cases} \textit{Water balances} \\ \textit{Topological constraints} \\ \textit{Water quality criteria} \end{cases}
$$

MOO classical treatment

Stand-alone structure

Enterprise		1	2	3	Total
Water flowrate (tonne/hr)	Fresh	98.33	54.64	186.67	339.64
Cost (MMUSD/year)	Freshwater+discharge	0.28	0.15	0.52	0.95
	Reused water	0.01	0.01	0.02	0.03
	Total	0.28	0.16	0.54	0.98

Eco-park structure : MOO approach

Enterprise		1	2	3	Total
Water flowrate (tonne/hr)	Fresh	88.33	20.00	206.02	314.36
	Shared	76.67	61.04	82.00	219.71
Cost (MMUSD/year)	Freshwater+ Discharge	0.18	0.11	0.59	0.88
	Reused water	0.01	0.02	0.02	0.06
	Total	0.20	0.13	0.61	0.94

Alternative approach

The needed change :

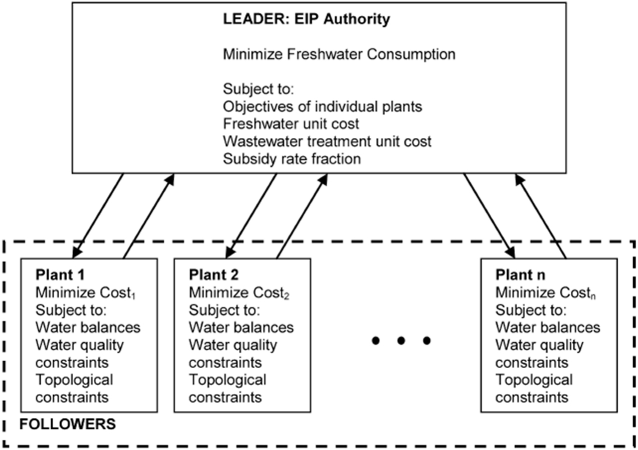
. . . to have an independant designer/regulator

. . . to have fair solutions for the companies

Thus we propose to use two different possible models:

- Hierarchical optimisation (bi-level optim.)
- Nash game concept between the companies

Single-Leader-Multi-Follower game



Numerical treatment

This very difficult problem is treated as follows:

- first we replace the lower-level (convex) optimization problem by their KKT systems; the resulting problem is an Mathematical Programming with Complementarity Constraints (MPCC);
- second the MPCC problem is solved by penalization methods

Numerical results have been obtained with Julia meta-solver coupled with Gurobi, IPOPT and Baron.

Single-Leader-Multi-Follower game

Stand-alone implementation with regeration units

Enterprise		1	2	3	Total
Water flowrate (tonne/hr)	Fresh	98.33	22.00	97.50	217.83
	Regenerated	0.00	38.17	111.46	149.63
Cost (MMUSD/year)	Freshwater+discharge	0.28	0.06	0.27	0.61
	Reused water	0.01	0.02	0.05	0.08
	Regenerated water	0.00	0.08	0.19	0.27
	Total	0.28	0.17	0.51	0.96

Nash equilibrium (SLMFG) with regeneration units

Enterprise		1	2	3	Total
Water flowrate (tonne/hr)	Freshwater (tonne/hr)	20.00	20.00	20.00	60.00
	Shared	126.49	149.54	226.66	502.69
	Regenerated	100.62	64.67	166.64	331.93
Cost (MMUSD/year)	Freshwater+Discharge	0.04	0.02	0.11	0.17
	Reused water	0.04	0.03	0.08	0.15
	Regenerated water	0.12	0.08	0.19	0.39

More on Single-Leader-Multi-Follower games

D.A & A. Svensson (J. Optim. Theory Appl. 182 (2019))

Existence for optimistic SLMF games

Theorem

Assume that F is lower semi-continuous, and for each follower $i = 1, ..., M$ the objective f_i is continuous and $(x, y_{-i}) \mapsto C_i(x, y_{-i}) := \{y_i \mid g_i(x, y) \leq 0\}$ is a lower semi-continuous set-valued map which has nonempty compact graph. If the graph of GNEP *is nonempty, then the SLMF game admits an optimistic solution.*

Example of linear pessimistic SLMF game with no solutions

Let us consider the SLMFG with two followers

$$\min_{x\in[0,4]} \max_{y\in GNEP(x)} -x + (y_1 + y_2).$$

with

$$\begin{array}{llcll} \min_{y_1} & y_1 & \qquad & \min_{y_2} & y_2 \\ s.t. & \left\{ \begin{array}{l} y_1 \geq 0 \\ 2y_2 - y_1 \leq 2 \\ y_1 + y_2 \geq x \end{array} \right. & & s.t. & \left\{ \begin{array}{l} y_2 \geq 0 \\ 2y_1 - y_2 \leq 2 \\ y_1 + y_2 \geq x \end{array} \right. \end{array}$$

The solution of the parametric GNEP of the followers is given by

$$\text{GNEP}(x) = \begin{cases} \{(0,0),(2,2)\} & \text{if } x \geq 4, \\ \{(0,0)\} & \text{if } x \in [0,4[\\ \emptyset & \text{otherwise.} \end{cases} \quad (5)$$

Notice that the function

$$\varphi_{max}(x) := \max_{y \in \text{GNEP}(x)} -x + (y_1 + y_2)$$
$$= \begin{cases} 0 & \text{if } x = 4 \\ -x & \text{if } x \in [0,4[\end{cases}$$

is not lower semi-continuous, so that Weierstrass theorem argument cannot be applied. And in fact, the value of the problem of the leader is -4, while there does not exist a point $x \in [0,4]$ with that value. The pessimistic linear single-leader-two-follower problem has no optimal solution.

Going back to applications: IEP

Another approach: the blind/control input models

In two very recent works we suggested some reformulations of the optimal design problem:

- under some hypothesis (unique process for each company, linearization in the case of regeneration units), we shown that the optimal design problem can be reformulated as a classical Mixed Integer Linear Programming problem (MILP);
- this problem can be treated with classical tools (CPLEX);

Moreover we inserted a "minimal gain" condition

$$\mathrm{Cost}_i(x_i, x^P_{-i}, x^R, E) \leq \alpha_i \cdot STC_i, \qquad \forall i \in I_P.$$

ensuring that each participating company will gain at minimum $\alpha\%$ on its production cost.

Another appraoch: the blind/control input models

Theorem

For $E \in \mathcal{E}$ and $x^R \geq 0$ fixed, the equilibrium set $\mathrm{Eq}(x^R, E)$ *is given by*

$$\mathrm{Eq}(x^R, E) = \left\{ x^P \;:\; \forall i \in I_P, \begin{array}{l} z_i(x_{-i}) + \sum_{(k,i)\in E} x_{k,i} = \sum_{(i,j)\in E} x_{i,j} \\ g_i(x_{-i}) \leq 0 \\ z_i(x_{-i}) \geq 0 \\ x_i\big|_{E^c_{i,\mathrm{act}}} = 0 \\ x_i \geq 0 \end{array} \right\} \tag{6}$$

Thus, the optimal design problem is equivalent to

$$\min_{E\in\mathcal{E},\, x\in\mathbb{R}^{|E_{\max}|}} Z(x)$$

$$s.t. \begin{cases} x \in X, & \\ z_i(x_{-i}) + \sum_{(k,i)\in E} x_{k,i} = + \sum_{(i,j)\in E} x_{i,j}, & \forall i \in I \\ x_i\big|_{E^c_{i,\mathrm{act}}} = 0, & \forall i \in I \\ g_i(x_{-i}) \leq 0, & \forall i \in I \\ z_i(x_{-i}) \geq 0, & \forall i \in I \\ \mathrm{Cost}_i(x_i, x^P_{-i}, x^R, E) \leq \alpha_i \cdot STC_i, & \forall i \in I_P \\ x \geq 0. & \end{cases} \tag{7}$$

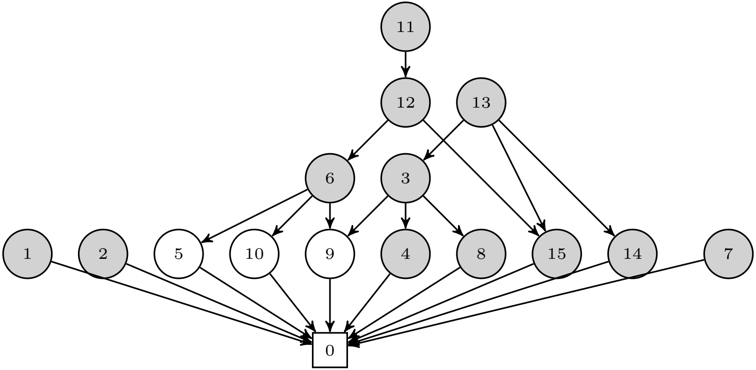


Figure: The configuration in the case without regeneration units, $\alpha_i = 0.95$ and Coef $= 1$. Gray nodes are consuming strictly positive fresh water.

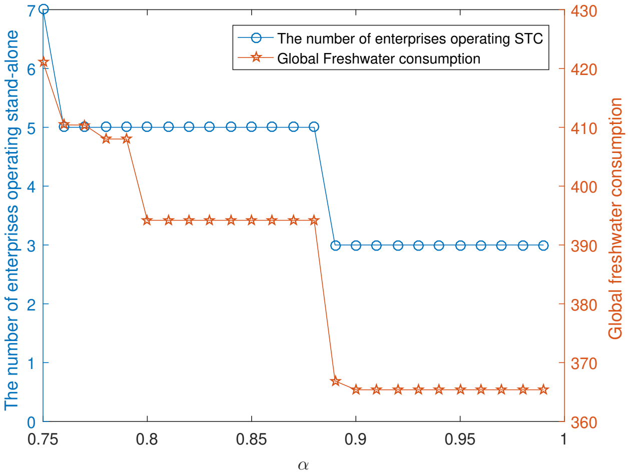


Figure: The number of enterprises operating stand-alone and the global freshwater consumption with Coef = 1.

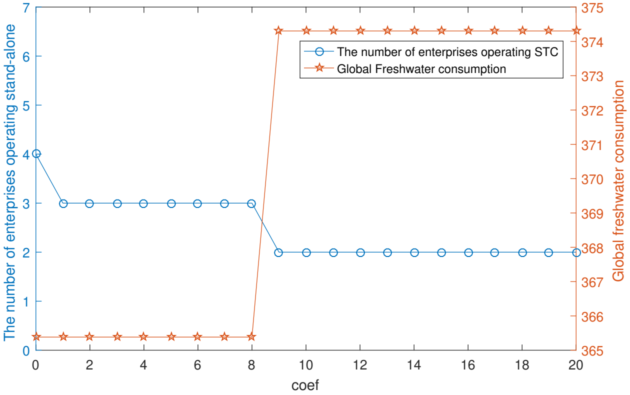


Figure: The number of enterprises operating stand-alone and the global freshwater consumption with $\alpha = 0.99$.

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MLFG in the setting of quasiconvex optimization

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ALOP autumn school - October 14th, 2020

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I - Introduction to quasiconvex optimization

Quasiconvexity

- A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *quasiconvex* on K if,

for all $x, y \in K$ *and all* $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}.$$

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or

for all $\lambda \in \mathbb{R}$, the sublevel set

$$S_\lambda = \{x \in X \;:\; f(x) \leq \lambda\} \textit{ is convex.}$$

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or

f differentiable

$$f \text{ is quasiconvex} \iff df \text{ is quasimonotone}$$

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for all $x, y \in K$ *and all* $t \in [0,1]$,

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}.$$

or

for all $\lambda \in \mathbb{R}$, *the sublevel set*

$$S_\lambda = \{x \in X \ : \ f(x) \leq \lambda\} \text{ is convex.}$$

or

f *differentiable*

$$f \text{ is quasiconvex} \iff df \text{ is quasimonotone}$$

or

$$f \text{ is quasiconvex} \iff \partial f \text{ is quasimonotone}$$

Quasiconvexity

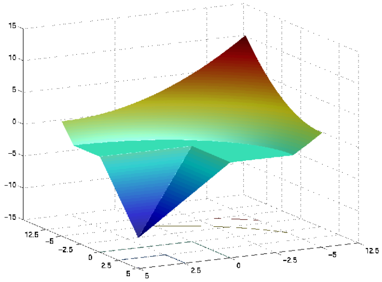
- A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *quasiconvex* on K if,

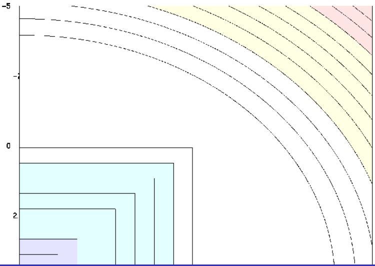
for all $\lambda \in \mathbb{R}$, the sublevel set

$$S_\lambda = \{x \in X \ : \ f(x) \leq \lambda\} \textit{ is convex.}$$

- A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *semistrictly quasiconvex* on K if, f is quasiconvex and for any $x, y \in K$,

$$f(x) < f(y) \Rightarrow f(z) < f(y), \quad \forall z \in [x, y[.$$





Why not a subdifferential for quasiconvex programming?

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- No (upper) semicontinuity of ∂f if f is not supposed to be Lipschitz

Why not a subdifferential for quasiconvex programming?

- No (upper) semicontinuity of ∂f if f is not supposed to be Lipschitz
- No sufficient optimality condition

$$\bar{x} \in S_{str}(\partial f, C) \not\Longrightarrow \bar{x} \in \arg\min_C f$$

II - Normal approach

a- First definitions

A first approach

Sublevel set:

$$S_\lambda = \{x \in X \ : \ f(x) \le \lambda\}$$
$$S_\lambda^> = \{x \in X \ : \ f(x) < \lambda\}$$

Normal operator:

Define $N_f(x) : X \to 2^{X^*}$ by

$$\begin{aligned} N_f(x) &= N(S_{f(x)}, x) \\ &= \{x^* \in X^* \ : \ \langle x^*, y - x\rangle \le 0, \ \ \forall\, y \in S_{f(x)}\}. \end{aligned}$$

With the corresponding definition for $N_f^>(x)$

But ...

- $N_f(x) = N(S_{f(x)}, x)$ has no upper-semicontinuity properties
- $N_f^{>}(x) = N(S_{f(x)}^{>}, x)$ has no quasimonotonicity properties

But ...

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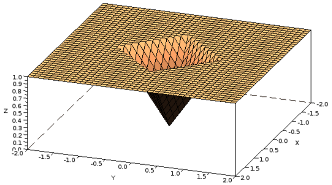
Example
Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(a, b) = \begin{cases} |a| + |b|, & \text{if } |a| + |b| \leq 1 \\ 1, & \text{if } |a| + |b| > 1 \end{cases}.$$



Then f is quasiconvex.

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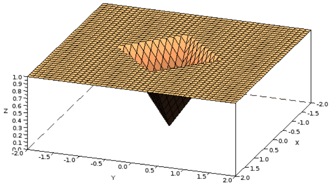


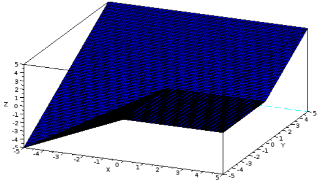
Then f is quasiconvex.
Consider $x = (10, 0)$, $x^* = (1, 2)$, $y = (0, 10)$ and $y^* = (2, 1)$.

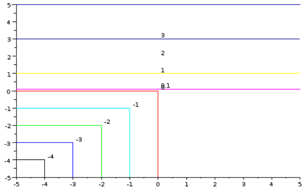
We see that $x^* \in N^<(x)$ and $y^* \in N^<(y)$ (since $|a| + |b| < 1$ implies $(1, 2) \cdot (a - 10, b) \leq 0$ and $(2, 1) \cdot (a, b - 10) \leq 0$) while $\langle x^*, y - x \rangle > 0$ and $\langle y^*, y - x \rangle < 0$. Hence $N^<$ is not quasimonotone.

But ...another example

- $N_f(x) = N(S_{f(x)}, x)$ has no upper-semicontinuity properties
- $N_f^{>}(x) = N(S^{>}_{f(x)}, x)$ has no quasimonotonicity properties

Example





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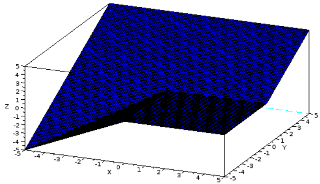

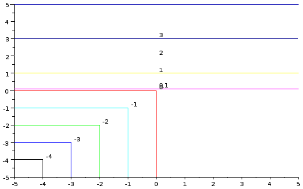

Then f is quasiconvex.

We easily see that $N(x)$ is not upper semicontinuous....

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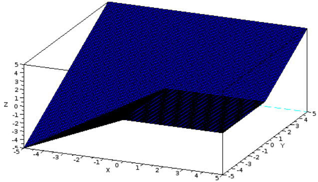
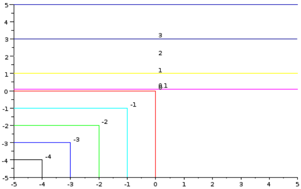
- $N_f^>(x) = N(S^>_{f(x)}, x)$ has no quasimonotonicity properties

Example





Then f is quasiconvex.

We easily see that $N(x)$ is not upper semicontinuous....

These two operators are essentially adapted to the class of semi-strictly quasiconvex functions. Indeed in this case, for each $x \in \mathrm{dom}\, f \setminus \arg\min f$,

II - Normal approach

b- Adjusted sublevel sets and normal operator

Definition

Adjusted sublevel set

For any $x \in \operatorname{dom} f$, we define

$$S_f^a(x) = S_{f(x)} \cap \overline{B}(S_{f(x)}^<, \rho_x)$$

where $\rho_x = dist(x, S_{f(x)}^<)$, if $S_{f(x)}^< \neq \emptyset$

and $S_f^a(x) = S_{f(x)}$ if $S_{f(x)}^< = \emptyset$.

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  e.g. f is semistrictly quasiconvex

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Proposition

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be any function, with domain $\operatorname{dom} f$. Then

$$f \textit{ is quasiconvex } \iff S_f^a(x) \textit{ is convex }, \forall\, x \in \operatorname{dom} f.$$

Adjusted normal operator

Adjusted sublevel set:

For any $x \in \mathrm{dom}\, f$, we define

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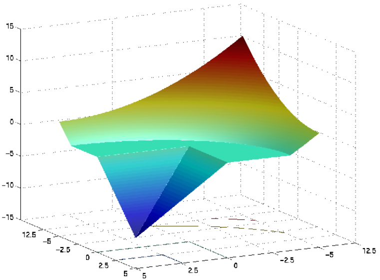
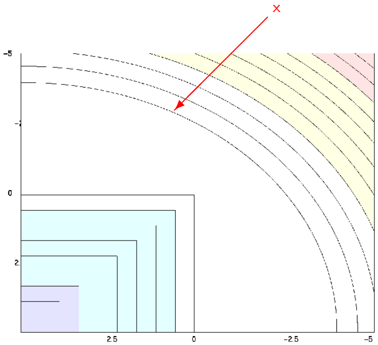
Ajusted normal operator:

$$N_f^a(x) = \{x^* \in X^* \;\;:\;\; \langle x^*, y - x\rangle \leq 0, \;\; \forall\, y \in S_f^a(x)\}$$

Example





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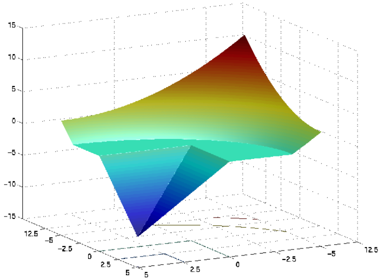
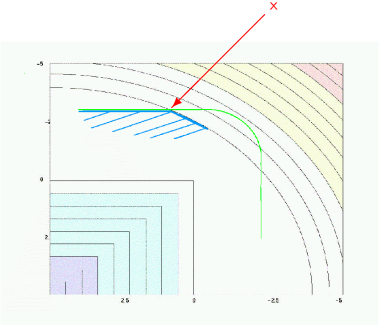
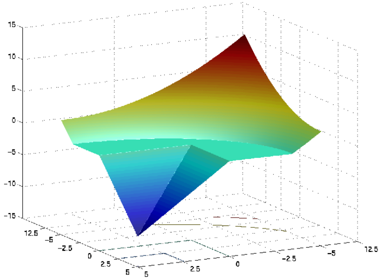


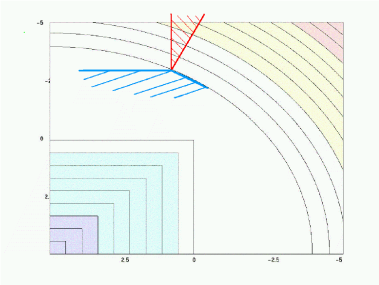


$\overline{B}(S^{<}_{f(x)}, \rho_x)$

$S^a_f(x) = S_f(x) \cap \overline{B}(S^{<}_{f(x)}, \rho_x)$

Example



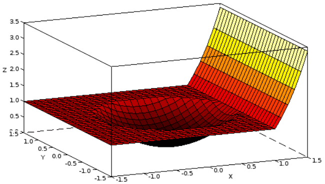


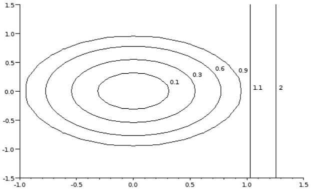
$$S_f^a(x) = S_f(x) \cap \overline{B}(S^{<}_{f(x)}, \rho_x)$$

$$N_f^a(x) = \{x^* \in X^* \; : \; \langle x^*, y - x\rangle \leq 0, \quad \forall\, y \in S_f^a(x)\}$$

An exercice.........

Let us draw the normal operator value $N_f^a(x, y)$ at the points $(x, y) = (0.5, 0.5)$, $(x, y) = (0, 1)$, $(x, y) = (1, 0)$, $(x, y) = (1, 2)$, $(x, y) = (1.5, 0)$ and $(x, y) = (0.5, 2)$.





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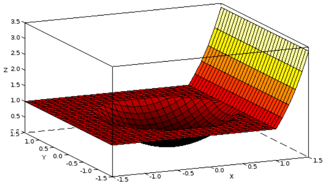

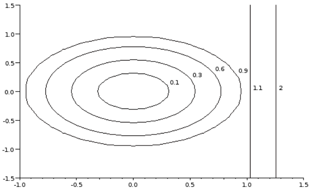

Operator N_f^a provide information at any point!!!

Basic properties of N_f^a

Nonemptyness:

Proposition

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ *be lsc. Assume that rad. continuous on* $\operatorname{dom} f$ *or* $\operatorname{dom} f$ *is convex and* $\operatorname{int} S_\lambda \neq \emptyset$, $\forall\, \lambda > \inf_X f$. *Then*

f *is quasiconvex* $\Leftrightarrow N_f^a(x) \setminus \{0\} \neq \emptyset, \quad \forall\, x \in \operatorname{dom} f \setminus \arg\min f$.

Quasimonotonicity:

The normal operator N_f^a is always quasimonotone

Upper sign-continuity

- $T : X \to 2^{X^*}$ is said to be *upper sign-continuous* on K iff for any $x, y \in K$, one have :

$$\forall\, t \in]0,1[, \quad \inf_{x^* \in T(x_t)} \langle x^*, y - x\rangle \geq 0 \implies \sup_{x^* \in T(x)} \langle x^*, y - x\rangle \geq 0$$

where $x_t = (1-t)x + ty$.

upper semi-continuous

$\Downarrow$

upper hemicontinuous

$\Downarrow$

upper sign-continuous

locally upper sign continuity

Definition

Let $T : K \to 2^{X^*}$ *be a set-valued map.*

T *is called locally upper sign-continuous on* K *if, for any* $x \in K$ *there exist a neigh.* V_x *of* x *and a upper sign-continuous set-valued map* $\Phi_x(\cdot) : V_x \to 2^{X^*}$ *with nonempty convex* w^**-compact values such that* $\Phi_x(y) \subseteq T(y) \setminus \{0\}, \forall\, y \in V_x$

locally upper sign continuity

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Continuity of normal operator

Proposition

Let f be lsc quasiconvex function such that int$(S_\lambda) \neq \emptyset, \forall\, \lambda > \inf f$.

Then N^a_f is locally upper sign-continuous on $\mathrm{dom}\, f \setminus \arg\min f$.

III

Quasiconvex programming

a- Optimality conditions

Quasiconvex programming

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ and $K \subseteq \operatorname{dom} f$ be a convex subset.

$$(P) \qquad \text{find } \bar{x} \in K \;:\; f(\bar{x}) = \inf_{x \in K} f(x)$$

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Perfect case: f convex

$f : X \to \mathbb{R} \cup \{+\infty\}$ a proper convex function

K a nonempty convex subset of X, $\bar{x} \in K$ + C.Q.

Then

$$f(\bar{x}) = \inf_{x \in K} f(x) \quad \Longleftrightarrow \quad \bar{x} \in S_{str}(\partial f, K)$$

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What about f quasiconvex case?

$$\bar{x} \in S_{str}(\partial f(\bar{x}), K) \not\Longrightarrow \bar{x} \in \arg\min_K f$$

Sufficient optimality condition

Theorem

$f : X \to \mathbb{R} \cup \{+\infty\}$ quasiconvex, radially cont. on $\operatorname{dom} f$

$C \subseteq X$ such that $conv(C) \subset \operatorname{dom} f$.
Suppose that $C \subset int(\operatorname{dom} f)$ or $AffC = X$.

Then $\bar{x} \in S(N_f^a \setminus \{0\}, C) \quad \Longrightarrow \quad \forall x \in C, f(\bar{x}) \leq f(x)$.

where $\bar{x} \in S(N_f^a \setminus \{0\}, K)$ means that there exists $\bar{x}^* \in N_f^a(\bar{x}) \setminus \{0\}$ such that

$$\langle \bar{x}^*, c - x \rangle \geq 0, \qquad \forall c \in C.$$

Necessary and Sufficient conditions

Proposition

Let C be a closed convex subset of X, $\bar{x} \in C$ and $f : X \to \mathbb{R}$ be continuous semistrictly quasiconvex such that $\mathrm{int}(S_f^a(\bar{x})) \neq \emptyset$ *and* $f(\bar{x}) > \inf_X f$.
Then the following assertions are equivalent:

i) $f(\bar{x}) = \min_C f$

ii) $\bar{x} \in S_{str}(N_f^a \setminus \{0\}, C)$

iii) $0 \in N_f^a(\bar{x}) \setminus \{0\} + NK(C, \bar{x})$.

GNEP reformulation in quasiconvex case

To simplify the notations, we will denote, for any i and any $x \in \mathbb{R}^n$, by $S_i(x)$ and $A_i(x^{-i})$ the subsets of $\mathbb{R}^{n_i}$

$$S_i(x) = S^a_{\theta_i(\cdot,x^{-i})}(x^i) \quad \textit{and} \quad A_i(x^{-i}) = \arg\min_{\mathbb{R}^{n_i}} \theta_i(\cdot, x^{-i}).$$

In order to construct the variational inequality problem we define the following set-valued map $N^a_\theta : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ which is described,

for any $x = (x^1, \dots, x^p) \in \mathbb{R}^{n_1} \times \dots \times \mathbb{R}^{n_p}$, *by*

$$N^a_\theta(x) = F_1(x) \times \dots \times F_p(x),$$

$$\textit{where } F_i(x) = \begin{cases} \overline{B}_i(0,1) & \textit{if } x^i \in A_i(x^{-i}) \\ \mathrm{co}(N^a_{\theta_i}(x^i) \cap S_i(0,1)) & \textit{otherwise} \end{cases}$$

The set-valued map N^a_θ has nonempty convex compact values.

Sufficient condition

In the following we assume that X is a given nonempty subset X of $\mathbb{R}^n$, such that for any i, the set $X_i(x^{-i})$ is given as

$$X_i(x^{-i}) = \{x^i \in \mathbb{R}^{n_i} : (x^i, x^{-i}) \in X\}.$$

Theorem

Let us assume that, for any i, the function θ_i is continuous and quasiconvex with respect to the i-th variable. Then every solution of $S(N^a_\theta, X)$ is a solution of the GNEP.

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Note that the link between GNEP and variational inequality is valid even if the constraint set X is neither convex nor compact.

Lemma

Let $i \in \{1, \dots, p\}$. If the function θ_i is continuous quasiconvex with respect to the i-th variable, then,

$$0 \in F_i(\bar{x}) \iff \bar{x}^i \in A_i(\bar{x}^{-i}).$$

Proof. It is sufficient to consider the case of a point $\bar{x}$ such that $\bar{x}^i \notin A_i(\bar{x}^{-i})$. Since $\theta_i(\cdot, \bar{x}^{-i})$ is continuous at $\bar{x}^i$, the interior of $S_i(\bar{x})$ is nonempty. Let us denote by K_i the convex cone

$$K_i = N^a_{\theta_i}(\bar{x}^i) = (S_i(\bar{x}) - \bar{x}^i)^\circ.$$

By quasiconvexity of θ_i, K_i is not reduced to $\{0\}$. Let us first observe that, since $S_i(\bar{x})$ has a nonempty interior, K_i is a pointed cone, that is $K_i \cap (-K_i) = \{0\}$.
Now let us suppose that $0 \in F_i(\bar{x})$. By Caratheodory theorem, there exist vectors $v_i \in [K_i \cap S_i(0,1)]$, $i = 1, \dots, n+1$ and scalars $\lambda_i \geq 0$, $i = 1, \dots, n+1$ with

$$\sum_{i=1}^{n+1} \lambda_i = 1 \text{ and } 0 = \sum_{i=1}^{n+1} \lambda_i \, v_i.$$

Since there exists at least one $r \in \{1, \ldots, n+1\}$ such that $\lambda_r > 0$ we have

$$v_r = -\sum_{i=1, i\neq r}^{n+1} \frac{\lambda_i}{\lambda_r} v_i$$

which clearly shows that v_r is an element of the convex cone $-K_i$. But $v_r \in S_i(0,1)$ and thus $v_r \neq 0$. This contradicts the fact that K_i is pointed and the proof is complete. ■

Proof of necessary condition

Proof. Let us consider $\bar{x}$ to be a solution of $S(N^a_\theta, X)$. There exists $v \in N^a_\theta(\bar{x})$ such that

$$\langle v, y - \bar{x} \rangle \geq 0, \quad \forall\, y \in X. \qquad (*)$$

Let $i \in \{1, \dots, p\}$.
If $\bar{x}^i \in A_i(\bar{x}^{-i})$ then obviously $\bar{x}^i \in Sol_i(\bar{x}^{-i})$.
Otherwise $v^i \in F_i(\bar{x}) = \mathrm{co}(N^a_{\theta_i}(\bar{x}^i) \cap S_i(0,1))$. Thus, according to Lemma 2, there exist $\lambda > 0$ and $u^i \in N^a_{\theta_i}(\bar{x}^i) \setminus \{0\}$ satisfying $v^i = \lambda u^i$.
Now for any $x^i \in X_i(\bar{x}^{-i})$, consider $y = \left(\bar{x}^1, \dots, \bar{x}^{i-1}, x^i, \bar{x}^{i+1}, \dots, \bar{x}^p\right)$.
From $(*)$ one immediately obtains that $\langle u^i, x^i - \bar{x}^i \rangle \geq 0$. Since x^i is an arbitrary element of $X_i(\bar{x}^{-i})$, we have that $\bar{x}^i$ is a solution of $S(N^a_{\theta_i} \setminus \{0\}, X_i(\bar{x}^{-i}))$ and therefore, according to Prop. 4,

$$\bar{x}^i \in Sol_i(\bar{x}^{-i})$$

Since i was arbitrarily chosen we conclude that $\bar{x}$ solves the GNEP. ■

Necessary and sufficient condition

Theorem

Let us suppose that, for any i, the loss function θ_i is continuous and semistrictly quasiconvex with respect to the i-th variable. Further assume that the set X is a nonempty convex subset of $\mathbb{R}^N$. Then

any solution of the variational inequality $S(N^a_\theta, X)$ is a solution of the GNEP

any solution of the GNEP is a solution of the quasi-variational inequality $QVI(N^a_\theta, \mathcal{X})$

and
where $\mathcal{X}$ stands for the set-valued map defined on $\mathbb{R}^2$ by

$$\mathcal{X}(x) = \prod_{i=1}^{p} X_i(x^{-i})$$

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D.A. & J. Dutta, Oper. Res. Letters, 2008.