Bilevel Problems, MPCCs, and Multi-Leader-Follower Games

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Perpignan, France
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- Professor in Applied Mathematics at Univ. of Perpignan
- **Research topics:**
  - Bilevel programming, Nash games and in particular Multi-leader-follower games
  - Energy management: Electricity markets, Industrial Eco-Parks (IEP), Demand-side management
  - Variational and quasi-variational inequalities
  - Quasiconvex optimization

Research lab.: PROMES (CNRS)
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What do I work on?
What do I work on?

Optimization / Math. programming
What do I work on?

Optimization / Math. programming

Bilevel optimization
What do I work on?

Optimization / Math. programming

Nash games
What do I work on?

Optimization / Math. programming

- Bilevel optimization
- Nash games
- Multi-Leader-Follower games
Bilevel: some general comments
A Bilevel Problem consists in an upper-level/leader’s problem

\[
\text{“min}_{x \in \mathbb{R}^n} \quad F(x, y) \quad \text{s.t.} \quad \begin{cases}
  x \in X \\
  y \in S(x)
\end{cases}
\]

where \( \emptyset \neq X \subset \mathbb{R}^n \) and \( S(x) \) stands for the solution set of its lower-level/follower’s problem

\[
\min_{y \in \mathbb{R}^m} \quad f(x, y) \quad \text{s.t.} \quad g(x, y) \leq 0
\]
A trivial example

Consider the following simple bilevel problem

\[
\begin{align*}
&\text{``min}_{x \in \mathbb{R}}'' \quad x \\
&\text{s.t.} \quad \left\{ \begin{array}{l}
x \in [-1, 1] \\
y \in S(x)
\end{array} \right. 
\end{align*}
\]

with \( S(x) = \text{``y solving} \)

\[
\begin{align*}
\min_{y \in \mathbb{R}} \quad &-xy \\
\text{s.t.} \quad &x^2(y^2 - 1) \leq 0
\end{align*}
\]
Lower level problem:

\[
\begin{align*}
\min_{y \in \mathbb{R}} & \quad -x.y \\
\text{s.t} & \quad x^2(y^2 - 1) \leq 0
\end{align*}
\]

Note that the solution map of this convex problem is

\[
S(x) := \begin{cases} 
\{1\} & x < 0 \\
\{-1\} & x > 0 \\
\mathbb{R} & x = 0
\end{cases}
\]

Thus for each \( x \neq 0 \) there is a unique associated solution of the lower level problem
A trivial example

Lower level problem:

\[
\begin{align*}
\min_{y \in \mathbb{R}} & \quad -xy \\
\text{s.t.} & \quad x^2(y^2 - 1) \leq 0
\end{align*}
\]

Note that the solution map of this convex problem is
A trivial example

Consider the following simple bilevel problem

$$\min_{x \in \mathbb{R}} \quad -x.y$$

s.t.  

\[
\begin{align*}
\{ & \quad x \in [-1, 1] \\
\{ & \quad y \in S(x) \\
\end{align*}
\]

with \( S(x) = \) “\( y \) solving

\[
S(x) := \begin{cases}
\{1\} & x < 0 \\
\{-1\} & x > 0 \\
\mathbb{R} & x = 0
\end{cases}
\]
An *Optimistic Bilevel Problem* consists in an upper-level/leader’s problem

\[
\min_{x \in \mathbb{R}^n} \min_{y \in \mathbb{R}^m} \quad F(x, y) \\
\text{s.t.} \quad x \in X \\
\quad y \in S(x)
\]

where \( \emptyset \neq X \subset \mathbb{R}^n \) and \( S(x) \) stands for the solution set of its lower-level/follower’s problem

\[
\min_{y \in \mathbb{R}^m} \quad f(x, y) \\
\text{s.t.} \quad g(x, y) \leq 0
\]
An *Pessimistic Bilevel Problem* consists in an upper-level/leader’s problem

\[
\min_{x \in \mathbb{R}^n} \max_{y \in \mathbb{R}^m} \quad F(x, y) \\
\text{s.t.} \quad \begin{cases} 
  x \in X \\
  y \in S(x)
\end{cases}
\]

where $\emptyset \neq X \subset \mathbb{R}^n$ and $S(x)$ stands for the solution set of its lower-level/follower’s problem

\[
\min_{y \in \mathbb{R}^m} \quad f(x, y) \\
\text{s.t.} \quad g(x, y) \leq 0
\]
And of course the "confortable situation" corresponds to the case of a unique response

$$\forall x \in X, \quad S(x) = \{y(x)\}.$$ 

Then

$$\min_{x \in \mathbb{R}^n} \quad F(x, y(x))$$

s.t. $$\left\{ x \in X \right\}$$
And of course the "confortable situation" corresponds to the case of a unique response

$$\forall x \in X, \quad S(x) = \{y(x)\}.$$ 

Then

$$\min_{x \in \mathbb{R}^n} \quad F(x, y(x))$$

s.t. \quad \{ x \in X \}

For example when 

for any $x$, $g(x, \cdot)$ is quasiconvex and $f(x, \cdot)$ is strictly convex.
An "Selection-type" Bilevel Problem consists in an upper-level/leader’s problem

$$\min_{x \in \mathbb{R}^n} F(x, y(x))$$

s.t. \[
\begin{cases}
    x \in X \\
    y(x) \text{ is a uniquely determined selection of } S(x)
\end{cases}
\]

In one of the Elevator pitches (Monday), D. Salas and A. Svensson proposed a **probabilistic approach**:

- *Consider a probability on the different possible follower’s reactions*
- *Minimize the expectation of the leader(s)*
An alternative point of view

Instead of considering the previous (optimistic) formulation of BL:

\[
\min_{x \in \mathbb{R}^n} \min_{y \in \mathbb{R}^m} F(x, y) \\
\text{s.t.} \quad \begin{cases} 
  x \in X \\
  y \in S(x) 
\end{cases}
\]
An alternative point of view

Instead of considering the previous (optimistic) formulation of BL:

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} \min_{y \in \mathbb{R}^m} & \quad F(x, y) \\
\text{s.t.} & \quad \begin{cases}
  x \in X \\
  y \in S(x)
\end{cases}
\end{align*}
\]

one can define the (optimistic) value function

\[
\varphi_{\min}(x) = \min_y \{ F(x, y) : g(x, y) \leq 0 \} \tag{1}
\]

and the Bl problem becomes

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \quad \varphi_{\min}(x) \\
\text{s.t.} & \quad x \in X
\end{align*}
\]
An alternative point of view

Instead of considering the previous (pessimistic) formulation of BL:

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} \max_{y \in \mathbb{R}^m} & \quad F(x, y) \\
\text{s.t.} & \quad \begin{cases} 
\quad x \in X \\
\quad y \in S(x)
\end{cases}
\end{align*}
\]
An alternative point of view

Instead of considering the previous (pessimistic) formulation of BL:

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \ \max_{y \in \mathbb{R}^m} \quad F(x, y) \\
\text{s.t.} & \quad \begin{cases}
x \in X \\
y \in S(x)
\end{cases}
\end{align*}
\]

one can define the (pessimistic) value function

\[
\varphi_{\max}(x) = \max_y \{ F(x, y) : g(x, y) \leq 0 \}
\]

and the Bl problem becomes

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} & \ \varphi_{\max}(x) \\
\text{s.t.} & \quad x \in X
\end{align*}
\]
An alternative point of view

This is the point of view presented in Stephan Dempe’s book:

\[
\begin{align*}
\min_{x \in \mathbb{R}^n} \min \left/ \max_{y \in \mathbb{R}^m} \right. \quad & F(x, y) \\
\text{s.t.} \quad & \left\{ \begin{array}{l}
x \in X \\ y \in S(x)
\end{array} \right. \\
\end{align*}
\]

vs

\[
\min_{x \in \mathbb{R}^n} \varphi_{\min/\max}(x) \\
\text{s.t.} \quad & x \in X
\]

It immediately raises the question: **What is a solution?**

- An optimal leader’s optimal strategy?
- An optimal couple \((x, y)\) = couple of strategies of leader and follower?

Didier Aussel

Bilevel Problems, MPCCs, and Multi-Leader-Follower Games
An alternative point of view

This is the point of view presented in Stephan Dempe’s book:

\[
\min_{x \in \mathbb{R}^n} \min \left/ \max_{y \in \mathbb{R}^m} \right. \quad F(x, y) \quad \text{s.t.} \quad \begin{cases} 
  x \in X \\
  y \in S(x)
\end{cases} \quad \text{vs} \quad \min_{x \in \mathbb{R}^n} \varphi_{\min/\max}(x) \quad \text{s.t.} \quad x \in X
\]

It immediately raises the question

*What is a solution??*
An alternative point of view

This is the point of view presented in Stephan Dempe’s book:

\[
\min_{x \in \mathbb{R}^n} \min / \max_{y \in \mathbb{R}^m} F(x, y) \quad \text{s.t.} \quad \begin{cases} x \in X \\ y \in S(x) \end{cases} \quad \text{vs} \quad \min_{x \in \mathbb{R}^n} \phi \min/\max(x) \quad \text{s.t.} \quad x \in X
\]

It immediately raises the question

**What is a solution??**

- **an optimal** \(x = \text{leader's optimal strategy?} \)
- **an optimal couple** \((x, y) = \text{couple of strategies of leader and follower?}\)
Actually usually when considering BL

\[
\min_{x \in \mathbb{R}^n} \min_{y \in \mathbb{R}^m} F(x, y) \quad \text{s.t.} \quad \left\{ \begin{array}{l}
x \in X \\
y \in S(x) \end{array} \right.
\]

people say

- **Step A**: the leader plays first
- **Step B**: the follower reacts

But in real life it’s a little bit more complex....
Actually in real life, when considering BL

$$\min_{x \in \mathbb{R}^n} \min_{y \in \mathbb{R}^m} F(x, y) \quad \text{s.t.} \quad \begin{cases} x \in X \\ y \in S(x) \end{cases}$$

We only work for the leader!!
Actually in real life, when considering BL

$$\min_{x \in \mathbb{R}^n} \min_{y \in \mathbb{R}^m} \quad F(x, y)$$

\[ s.t. \begin{cases} x \in X \\ y \in S(x) \end{cases} \]

We only work for the leader!! Indeed

- the leader has a model of the follower’s reaction: optimistic or pessimistic and
- Step 1: we compute a solution $x$ or $(x, y)$ of the BL model
- Step 2: the leader plays $x$
- Step 3: the follower decides to play...whatever he wants!!!
An existence result (optimistic)

Definition

The Mangasarian-Fromovitz constraint qualification (MFCQ) is satisfied at \((\bar{x}, \bar{y})\) with \(\bar{y}\) feasible point of the problem

\[
\min_y \{ f(x, y) : g(x, y) \leq 0 \}
\]

if the system

\[
\nabla_y g_i(\bar{x}, \bar{y}) d < 0 \quad \forall \; i \in I(\bar{x}, \bar{y}) := \{ j : g_j(\bar{x}, \bar{y}) = 0 \}
\]

has a solution.
Assume that $X = \{ x \in \mathbb{R}^n : G(x) \leq 0 \}$

**Theorem (Bank, Guddat, Klatte, Kummer, Tammer (83))**

Let $\bar{x}$ with $G(\bar{x}) \leq 0$ be fixed.

- The set $\{ (x, y) : g(x, y) \leq 0 \}$ is not empty and compact;
- At each point $(\bar{x}, \bar{y}) \in \text{gph} S$ with $G(\bar{x}) \leq 0$, assumption (MFCQ) is satisfied;

then, the set-valued map $S(\cdot)$ is upper semicontinuous at $(\bar{x}, \bar{y})$ and the function $\varphi_o(\cdot)$ is continuous at $\bar{x}$. 
An existence result (cont.)

**Theorem**

Assume that

- the set \( \{(x, y) : g(x, y) \leq 0\} \) is not empty and compact;
- at each point \((\overline{x}, \overline{y}) \in \text{gph} S\) with \(G(\overline{x}) \leq 0\), assumptions (MFCQ) is satisfied;
- the set \( \{x : G(x) \leq 0\} \) is not empty and compact,

then optimistic bilevel problem has a (global) optimal solution.
Bilevel problems and MPCC reformulation
We consider a Bilevel Problem consisting in an upper-level / leader’s problem

$$\min_{x \in \mathbb{R}^n} F(x, y)$$

$$s.t. \ y \in S(x), \ x \in X$$

where $\emptyset \neq X \subset \mathbb{R}^n$, and $S(x)$ stands for the solution of its lower-level / follower’s problem

$$\min_{y \in \mathbb{R}^m} f(x, y)$$

$$s.t \ g(x, y) \leq 0$$

which we assume to be convex and smooth, i.e. $\forall x \in X$, the functions $f(x, \cdot)$ and $g_i(x, \cdot)$ are smooth convex functions, and the gradients $\nabla_y g_i, \nabla_y f$ are continuous, $i = 1, ..., p$. 
Replacing the lower-level problem by its KKT conditions, gives place to a Mathematical Program with Complementarity Constraints.

**Bilevel**

\[
\begin{align*}
\min_{x \in X} & \quad F(x, y) \\
\text{s.t.} & \quad y \in S(x)
\end{align*}
\]

with \( S(x) = \text{“y solving} \]

\[
\min_{y \in \mathbb{R}^m} & \quad f(x, y) \\
\text{s.t.} & \quad g(x, y) \leq 0
\]

**MPCC**

\[
\begin{align*}
\min_{x \in X} & \quad F(x, y) \\
\text{s.t.} & \quad (y, u) \in KKT(x)
\end{align*}
\]

with \( KKT(x) = \text{“(y, u) solving} \]

\[
\begin{align*}
\nabla_y f(x, y) + u^T \nabla_y g(x, y) &= 0 \\
0 &\leq u \perp -g(x, y) \geq 0
\end{align*}
\]

We write \( \Lambda(x, y) \) for the set of \( u \) satisfying \( (y, u) \in KKT(x) \).
Example 1

Consider the following Bilevel problem and its MPCC reformulation:

**Bilevel**

\[
\begin{align*}
\text{min} & \quad x \\
\text{s.t.} & \quad y \in S(x)
\end{align*}
\]

with \( S(x) = \text{“} y \text{ solving} \)

\[
\begin{align*}
\min & \quad xy \\
\text{s.t.} & \quad x^2(y^2 - 1) \leq 0
\end{align*}
\]

**MPCC**

\[
\begin{align*}
\text{min} & \quad x \\
\text{s.t.} & \quad (y, u) \in KKT(x)
\end{align*}
\]

with \( KKT(x) = \text{“} (y, u) \text{ solving} \)

\[
\begin{align*}
\begin{cases}
 x + u \cdot 2yx^2 = 0 \\
 0 \leq u \perp -x^2(y^2 - 1) \geq 0
\end{cases}
\end{align*}
\]

1. \((0, -1, u)\) is a local solution of “MPCC”, for any \( u \in \Lambda(0, -1) = \mathbb{R}_+ \)

2. \((0, -1)\) is NOT a local solution of “Bilevel”
(a) \((0, -1, u)\) is a local solution of MPCC, \(\forall u \in \mathbb{R}_+\).

(b) \((0, -1)\) isn’t a local solution of the Bilevel problem.
Optimistic and Pessimistic approaches

The optimistic Bilevel (OB) is

$$\min_{x} \min_{y} F(x, y)$$

s.t. $y \in S(x), x \in X.$

The pessimistic Bilevel (PB) is

$$\min_{x} \max_{y} F(x, y)$$

s.t. $y \in S(x), x \in X.$
Optimistic and Pessimistic approaches

The optimistic Bilevel (OB) is

\[
\min_x \min_y F(x, y) \\
\text{s.t. } y \in S(x), x \in X.
\]

The pessimistic Bilevel (PB) is

\[
\min_x \max_y F(x, y) \\
\text{s.t. } y \in S(x), x \in X.
\]

The optimistic MPCC (OMPCC):

\[
\min_x \min_y F(x, y) \\
\text{s.t. } (y, u) \in KKT(x), x \in X.
\]

The pessimistic MPCC (PMPCC):

\[
\min_x \max_y F(x, y) \\
\text{s.t. } (y, u) \in KKT(x), x \in X.
\]
Optimistic approach

Is bilevel programming a special case of a MPCC?

S. Dempe - J. Dutta (2012 Math. Prog.)

\[
\min_x \min_y F(x, y) \\
\text{s.t. } y \in S(x), x \in X.
\]
Local solutions for in optimistic approach

Definition

A local (resp. global) solution of (OB) is a point \((\bar{x}, \bar{y}) \in Gr(S)\) if there exists \(U \in N(\bar{x}, \bar{y})\) (resp. \(U = \mathbb{R}^n \times \mathbb{R}^m\)) such that

\[ F(\bar{x}, \bar{y}) \leq F(x, y), \quad \forall (x, y) \in U \cap Gr(S). \]

Definition

A local (resp. global) solution for (OMPCC) is a triplet \((\bar{x}, \bar{y}, \bar{u}) \in Gr(KKT)\) such that there exists \(U \in N(\bar{x}, \bar{y}, \bar{u})\) (resp. \(U = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p\)) with

\[ F(\bar{x}, \bar{y}) \leq F(x, y), \quad \forall (x, y, u) \in U \cap Gr(KKT). \]
In Dempe-Dutta it was considered the Slater type constraint qualification for a parameter $x \in X$:

**Slater:** $\exists y(x) \in \mathbb{R}^m$ s.t. $g_i(x, y(x)) < 0$, $\forall i = 1, \ldots, p$. 
Results for the optimistic case

Theorem 1 Dempe-Dutta (2012)

Assume the convexity condition and Slater’s CQ at \( \bar{x} \).

1. If \( (\bar{x}, \bar{y}) \) is a local solution for (OB), then for each \( \bar{u} \in \Lambda(\bar{x}, \bar{y}) \), \( (\bar{x}, \bar{y}, \bar{u}) \) is a local solution for (OMPCC).

2. Conversely, assume that Slater’s CQ holds on a neighbourhood of \( \bar{x} \), \( \Lambda(\bar{x}, \bar{y}) \neq \emptyset \), and \( (\bar{x}, \bar{y}, u) \) is a local solution of (OMPCC) for every \( u \in \Lambda(\bar{x}, \bar{y}) \). Then \( (\bar{x}, \bar{y}) \) is a local solution of (OB).
Under the convexity assumption and some CQ ensuring $KKT(x) \neq \emptyset$, $\forall x \in X$:

\[(\bar{x}, \bar{y}) \text{ sol of (OB)} \Rightarrow \forall \bar{u} \in \Lambda(\bar{x}, \bar{y}) \Rightarrow (\bar{x}, \bar{y}, \bar{u}) \text{ sol of (OMPCC)}\]

**Figure: Global solution comparison in optimistic approach**

\[(\bar{x}, \bar{y}) \text{ local sol of (OB)} \Rightarrow \forall \bar{u} \in \Lambda(\bar{x}, \bar{y}), (\bar{x}, \bar{y}, \bar{u}) \text{ local sol of (OMPCC)} \text{ if Slater's CQ holds around } \bar{x}\]

**Figure: Local solution comparison in optimistic approach**
Example 1 (optimistic)

Consider the following optimistic Bilevel problem

$$\min_{x \in [-1, 1]} \min_y x$$

s.t. $y \in S(x)$, $x \in \mathbb{R}$

with lower-level

$$\min_y -xy$$

s.t. $x^2(y^2 - 1) \leq 0$.

1. $(0, -1, u)$ is a local solution of (OMPCC), for any $u \in \Lambda(0, -1) = \mathbb{R}_+$
2. $(0, -1)$ is NOT a local solution of (OB).
Pessimistic Approach

Is bilevel programming a special case of a (MPCC)?


\[
\min_x \max_y F(x, y) \\
\text{s.t. } y \in S(x), x \in X.
\]
**Definition**

A pair \((\bar{x}, \bar{y})\) is said to be a *local (resp. global) solution* for (PB), if \((\bar{x}, \bar{y}) \in Gr(S_p)\) and \(\exists U \in N(\bar{x}, \bar{y})\) such that

\[
F(\bar{x}, \bar{y}) \leq F(x, y), \quad \forall (x, y) \in U \cap Gr(S_p).
\]

(3)

where \(S_p(x) := \text{argmax}_y \{F(x, y) | y \in S(x)\}\).

**Definition**

A triplet \((\bar{x}, \bar{y}, \bar{u})\) is said to be a *local (resp. global) solution* for (PMPCC), if \((\bar{x}, \bar{y}, \bar{u}) \in Gr(KKT_p)\) and \(\exists U \in N(\bar{x}, \bar{y}, \bar{u})\) such that

\[
F(\bar{x}, \bar{y}) \leq F(x, y, u), \quad \forall (x, y, u) \in U \cap Gr(KKT_p).
\]

(4)

where \(KKT_p(x) := \text{argmax}_{y,u} \{F(x, y) | (y, u) \in KKT(x)\}\).
Theorem 2

Assume the convexity condition and that $\text{KKT}(x) \neq \emptyset$, $\forall x \in X$.

1. If $(\bar{x}, \bar{y})$ is a local solution for (PB), then for each $\bar{u} \in \Lambda(\bar{x}, \bar{y})$, $(\bar{x}, \bar{y}, \bar{u})$ is a local solution for (PMPCC).

2. Conversely, assume that one of the following condition are satisfied:
   1. The multifunction $\text{KKT}_p$ is LSC around $(\bar{x}, \bar{y}, \bar{u})$ and $(\bar{x}, \bar{y}, \bar{u})$ is a local solution of (PB).
   2. Slater’s CQ holds on a neighbourhood of $\bar{x}$, $\Lambda(\bar{x}, \bar{y}) \neq \emptyset$, and for every $u \in \Lambda(\bar{x}, \bar{y})$, $(\bar{x}, \bar{y}, u)$ is a local solution of (PMPCC).

Then $(\bar{x}, \bar{y})$ is a local solution of (PB).
Example 1 (pessimistic)

Consider the following pessimistic bilevel problem

\[
\begin{align*}
&\min_{x \in [-1,1]} \max_{y} x \\
\text{s.t. } & y \in S(x), \; x \in \mathbb{R}
\end{align*}
\]

with lower-level

\[
\begin{align*}
&\min_{y} -xy \\
\text{s.t. } & x^2(y^2 - 1) \leq 0.
\end{align*}
\]

1. \((0, -1, u)\) is a local solution of (PMPCC), for any \(u \in \Lambda(0, -1) = \mathbb{R}_+\)

2. \((0, -1)\) is NOT a local solution of (PB).
Example 2

Consider the following Bilevel problem

$$\text{“min”}_x x$$

$$\text{s.t. } y \in S(x)$$

with $S(x)$ the solution of the lower-level problem

$$\min_y \{-y \mid x + y \leq 0, y \leq 0\}$$

Even though Slater’s CQ holds, we have

1. $(0, 0, u_1, u_2)$ with $(u_1, u_2) \in \Lambda(0, 0) = \{(\lambda, 1 - \lambda) \mid \lambda \in [0, 1]\}$ is a local solution of “(MPCC)”, iff $u_1 \neq 0$,

2. $(0, 0)$ is NOT a local solution for “(B)”. 
Under the convexity assumption and some (CQ) ensuring \( KKT(x) \neq \emptyset \), \( \forall x \in X \):

\[
\forall \bar{u} \in \Lambda(\bar{x}, \bar{y})
\]

\((\bar{x}, \bar{y})\) sol of (PB)

\((\bar{x}, \bar{y}, \bar{u})\) sol of (PMPCC)

**Figure:** Global solution comparison in pessimistic approach

\[
\forall \bar{u} \in \Lambda(\bar{x}, \bar{y}),
\]

\((\bar{x}, \bar{y}, \bar{u})\) local sol of (PMPCC)

Slater’s CQ for all \( x \) around \( \bar{x} \)

**Figure:** Local solutions comparison in pessimistic approach
An introduction to MLFG
A Nash equilibrium problem is a noncooperative game in which the decision function (cost/benefit) of each player depends on the decision of the other players.
A Nash equilibrium problem is a noncooperative game in which the decision function (cost/benefit) of each player depends on the decision of the other players.

Denote by $N$ the number of players and each player $i$ controls variables $x^i \in \mathbb{R}^{n_i}$. The “total strategy vector” is $x$ which will be often denoted by

$$x = (x^i, x^{-i}).$$

where $x^{-i}$ is the strategy vector of the other players.
The strategy of player $i$ belongs to a strategy set

$$x^i \in X_i$$
The strategy of player $i$ belongs to a strategy set

$$x^i \in X_i$$

Given the strategies $x^{-i}$ of the other players, the aim of player $i$ is to choose a strategy $x^i$ solving

$$P_i(x^{-i}) \quad \text{max} \quad \theta_i(x^i, x^{-i})$$

$$\text{s.t.} \quad x^i \in X_i$$

where $\theta_i(\cdot, x^{-i}) : \mathbb{R}^{n_i} \to \mathbb{R}$ is the decision function for player $i$. 
Nash Equilibrium Problem (NEP)

- The strategy of player $i$ belongs to a strategy set
  \[ x^i \in X_i \]

- Given the strategies $x^{-i}$ of the other players, the aim of player $i$ is to choose a strategy $x^i$ solving
  \[
  \begin{align*}
  P_i(x^{-i}) & \quad \max \quad \theta_i(x^i, x^{-i}) \\
  \text{s.t.} & \quad x^i \in X_i
  \end{align*}
  \]
  where $\theta_i(\cdot, x^{-i}) : \mathbb{R}^{n_i} \to \mathbb{R}$ is the decision function for player $i$.

- A vector $\bar{x}$ is a Nash Equilibrium if
  \[
  \text{for any } i, \quad \bar{x}^i \text{ solves } P_i(\bar{x}^{-i}).
  \]
A generalized Nash equilibrium problem (GNEP) is a noncooperative game in which the decision function and strategy set of each player depend on the decision of the other players.

The strategy of player $i$ belongs to a strategy set

$$x^i \in X_i(x^{-i})$$

which depends on the decision variables of the other players.
A generalized Nash equilibrium problem (GNEP) is a noncooperative game in which the decision function and strategy set of each player depend on the decision of the other players.

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$$\text{s.t.} \quad x^i \in X_i(x^{-i})$$

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The strategy of player $i$ belongs to a strategy set

$$x^i \in X_i(x^{-i})$$

which depends on the decision variables of the other players.

Given the strategies $x^{-i}$ of the other players, the aim of player $i$ is to choose a strategy $x^i$ solving

$$P_i(x^{-i}) \quad \text{max} \quad \theta_i(x^i, x^{-i})$$
$$\text{s.t.} \quad x^i \in X_i(x^{-i})$$

where $\theta_i(\cdot, x^{-i}) : \mathbb{R}^{ni} \rightarrow \mathbb{R}$ is the decision function for player $i$.

A vector $\bar{x}$ is a Generalized Nash Equilibrium if

for any $i$, $\bar{x}^i$ solves $P_i(\bar{x}^{-i})$. 

Didier Aussel
Generalized Nash game (GNEP):

\[
\begin{align*}
\min_{x_1} & \quad \theta_1(x) \\
\text{s.t.} & \quad \{ \ x_1 \in X_1(x_{-1}) \} \\
\end{align*}
\quad \ldots \quad
\begin{align*}
\min_{x_n} & \quad \theta_n(x) \\
\text{s.t.} & \quad \{ \ x_n \in X_n(x_{-n}) \}
\end{align*}
\]
A classical existence result

Theorem (Ichiishi-Quinzii 1983)

Let a GNEP be given and suppose that

1. For each $\nu = 1, \ldots, N$ there exist a nonempty, convex and compact set $K_\nu \subset \mathbb{R}^{n_\nu}$ such that the point-to-set map $X^\nu : K_{-\nu} \rightrightarrows K_\nu$, is both upper and lower semicontinuous with nonempty closed and convex values, where $K_{-\nu} := \prod_{\nu' \neq \nu} K_{\nu'}$.

2. For every player $\nu$, the function $\theta^\nu$ is continuous and $\theta^\nu (\cdot, x^{-\nu})$ is quasi-convex on $X^\nu (x^{-\nu})$.

Then a generalized Nash equilibrium exists.

Note that in Aussel-Dutta (2008) an alternative proof of existence of equilibria has been given, under the assumption of the Rosen’s law, by using the normal approach technique.
Structure of the set of GNEPs

Example

Let $x = (x^1, x^2) \in [0, 4]^2$ and $f^\nu(x) := d_{T^\nu}(x)^2$, where $T_1$ is the triangle with vertices $(0, 0)$, $(0, 4)$ and $(1, 2)$, and $T_2$ is the triangle whose vertices are $(0, 0)$, $(4, 0)$ and $(2, 1)$. Let $S^\nu(x^{-\nu}) := \arg\min_{x^\nu} \{ f^\nu(x^1, x^2) \mid x^\nu \in [0, 4] \}$. We see that

- $S_1(x^2) = \{ x^1 \in [0, 4] \mid (x^1, x^2) \in T_1 \}$ for $x^2 \in [0, 1]
- S_1(x^2) = \{2\}$ for all $x^2 \in (1, 4])$
- $S_2(x^1) = \{ x^2 \in [0, 4] \mid (x^1, x^2) \in T_2 \}$ for $x^1 \in [0, 1]$  
- $S_2(x^1) = \{2\}$ for all $x^1 \in (1, 4]$.
Structure of the set of GNEPs (cont.)

\[ x^1 \]

\[ S_1(\cdot) \]

\[ S_2(\cdot) \]
A particular case

Multi-Leader-Follower-Game (MLFG):

\[
\begin{align*}
\min_{x_1, y_1, \ldots, y_p} & \quad \theta_1(x, y) \\
\text{s.t.} & \quad \begin{cases} 
    x_1 \in X_1(x_{-1}) \\
    y \in Y(x)
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\min_{x_n, y_1, \ldots, y_p} & \quad \theta_n(x, y) \\
\text{s.t.} & \quad \begin{cases} 
    x_n \in X_n(x_{-n}) \\
    y \in Y(x)
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\min_{y_1, \ldots, y_p} & \quad \phi_1(x, y) \\
\text{s.t.} & \quad \begin{cases} 
    y \in Y(x)
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\min_{y_1, \ldots, y_p} & \quad \phi_p(x, y) \\
\text{s.t.} & \quad \begin{cases} 
    y \in Y(x)
\end{cases}
\end{align*}
\]
and another problem

**Single-Leader-Multi-Follower-Game (SLMFG):**

\[
\begin{align*}
\min_{x, y_1, \ldots, y_p} & \quad \theta_1(x, y) \\
\text{s.t.} & \quad \left\{ \begin{array}{l}
x \in X \\
y \in Y(x)
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
\min_{y_1, \ldots, y_p} & \quad \phi_1(x, y) \\
\text{s.t.} & \quad \left\{ y \in Y(x) \right.
\end{align*}
\]

\[
\begin{align*}
\min_{y_1, \ldots, y_p} & \quad \phi_p(x, y) \\
\text{s.t.} & \quad \left\{ y \in Y(x) \right.
\end{align*}
\]
A particular case

Multi-Leader-Single-Follower-Game (MLSFG):

\[
\begin{align*}
\min_{x_1, y_1, \ldots, y_p} & \quad \theta_1(x, y) \\
\text{s.t.} & \quad \begin{cases} 
  x_1 \in X_1(x_{-1}) \\
  y \in Y(x)
\end{cases}
\end{align*}
\]

\[\cdots\]

\[
\begin{align*}
\min_{x_n, y_1, \ldots, y_p} & \quad \theta_n(x, y) \\
\text{s.t.} & \quad \begin{cases} 
  x_n \in X_n(x_{-n}) \\
  y \in Y(x)
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\min_{y_1, \ldots, y_p} & \quad \phi_1(x, y) \\
\text{s.t.} & \quad \begin{cases} 
  y \in Y(x)
\end{cases}
\end{align*}
\]

Didier Aussel
\[
\min_{x_1, y} \quad \theta_1(x_1, x_2, y) = x_1 y \\
\text{s.t.} \quad \begin{cases} 
    x_1 \in [0, 1] \\
    y \in S(x_1, x_2) 
\end{cases}
\]

\[
\min_{x_2, y} \quad \theta_1(x_1, x_2, y) = -x_2 y \\
\text{s.t.} \quad \begin{cases} 
    x_2 \in [0, 1] \\
    y \in S(x_1, x_2) 
\end{cases}
\]

with

\[
\min_{y} \quad f(x_1, x_2, y) = \frac{1}{3} y^3 - (x_1 + x_2)^2 y \\
\text{s.t.} \quad y \in \mathbb{R}
\]
MLSF game: ill-posedness

\[
\begin{align*}
\min_{x_1,y} & \quad \theta_1(x_1, x_2, y) = x_1.y \\
\text{s.t.} & \quad \begin{cases} 
  x_1 \in [0, 1] \\
  y \in S(x_1, x_2)
\end{cases}
\end{align*}
\]

\[
\begin{align*}
\min_{x_2,y} & \quad \theta_1(x_1, x_2, y) = -x_2.y \\
\text{s.t.} & \quad \begin{cases} 
  x_2 \in [0, 1] \\
  y \in S(x_1, x_2)
\end{cases}
\end{align*}
\]

with

\[
\begin{align*}
\min_y & \quad f(x_1, x_2, y) = \frac{1}{3} y^3 - (x_1 + x_2)^2 y \\
\text{s.t.} & \quad y \in \mathbb{R}
\end{align*}
\]

Exercise: Please analyse this small example...
The follower problem first

\[
\begin{align*}
\min_y \quad & f(x_1, x_2, y) = \frac{1}{3} y^3 - (x_1 + x_2)^2 y \\
\text{s.t.} \quad & y \in \mathbb{R}
\end{align*}
\]
The follower problem first

\[
\min_y \quad f(x_1, x_2, y) = \frac{1}{3} y^3 - (x_1 + x_2)^2 y \\
\text{s.t.} \quad y \in \mathbb{R}
\]

The solution map of this follower problem is

\[ S(x_1, x_2) = \{ y_1 = x_1 + x_2, y_2 = -x_1 - x_2 \}. \]
The solution map of this follower problem is

\[ S(x_1, x_2) = \{ y_1 = x_1 + x_2, y_2 = -x_1 - x_2 \}. \]

The leader 1 problem

\[ \theta_1(x, y) = x_1.y = \begin{cases} 
  x_1^2 + x_1.x_2 & \text{if } y = y_1 \\
  -x_1^2 - x_1.x_2 & \text{if } y = y_2
\end{cases} \]
The solution map of this follower problem is

\[ S(x_1, x_2) = \{ y_1 = x_1 + x_2, y_2 = -x_1 - x_2 \}. \]

The leader 1 problem

\[ \theta_1(x, y) = x_1.y = \begin{cases} 
  x_1^2 + x_1.x_2 & \text{if } y = y_1 \\
  -x_1^2 - x_1.x_2 & \text{if } y = y_2 
\end{cases} \]

Thus the response function of player 1 is

\[ R_1(x_2) = \begin{cases} 
  \{0\} & \text{if } y = y_1 \text{ with a payoff } = 0 \\
  \{1\} & \text{if } y = y_2 \text{ with a payoff } = -1 - x_2 
\end{cases} \]
The solution map of this follower problem is

\[ S(x_1, x_2) = \{ y_1 = x_1 + x_2, y_2 = -x_1 - x_2 \}. \]

The leader 2 problem

\[ \theta_1(x, y) = -x_2.y = \begin{cases} 
-x_1^2 - x_1.x_2 & \text{if } y = y_1 \\
x_1^2 + x_1.x_2 & \text{if } y = y_2 
\end{cases} \]
The solution map of this follower problem is

\[ S(x_1, x_2) = \{ y_1 = x_1 + x_2, y_2 = -x_1 - x_2 \}. \]

The leader 2 problem

\[ \theta_1(x, y) = -x_2.y = \begin{cases} -x_1^2 - x_1.x_2 & \text{if } y = y_1 \\ x_1^2 + x_1.x_2 & \text{if } y = y_2 \end{cases} \]

Thus the response function of player 1 is

\[ \mathbb{R}_2(x_1) = \begin{cases} \{1\} & \text{if } y = y_1 \text{ with a payoff } = -1 - x_1 \\ \{0\} & \text{if } y = y_2 \text{ with a payoff } = 0 \end{cases} \]
So the Nash equilibrium will be $(x_1, x_2) = (1, 1)$ but...

$$\mathcal{R}_1(x_2) = \begin{cases} 
\{(0, y = y_1)\} & \text{with a payoff} = 0 \\
\{(1, y = y_2)\} & \text{with a payoff} = -1 - x_2
\end{cases}$$

$$\mathcal{R}_2(x_1) = \begin{cases} 
\{(1, y = y_1)\} & \text{with a payoff} = -1 - x_1 \\
\{(0, y = y_2)\} & \text{with a payoff} = 0
\end{cases}$$
So the Nash equilibrium will be \((x_1, x_2) = (1, 1)\) but....
For the Demand-side management, we recently introduced the Multi-Leader-Disjoint-Follower game
Let us consider a 2-leader-single-follower game:

\[
\begin{align*}
\min_{x_1, y_1} & \quad \frac{1}{2} x_1 + y_1 & \min_{x_2, y_2} & \quad -\frac{1}{2} x_2 - y_2 \\
\{ & \quad x_1 \in [0, 1] & \{ & \quad x_2 \in [0, 1] \\
& \quad y \in S(x_1, x_2) & & \quad y \in S(x_1, x_2)
\end{align*}
\]

where \( S(x_1, x_2) \) is the solution map of

\[
\min_{y \geq 0} \quad y(-1 + x_1 + x_2) + \frac{1}{2} y^2
\]
Let us consider a 2-leader-single-follower game:

\[
\begin{align*}
\min_{x_1, y_1} & \quad \frac{1}{2} x_1 + y_1 \\
\{ & \quad x_1 \in [0, 1] \\
\} & \quad y_1 \in S(x_1, x_2) \\
\min_{x_2, y_2} & \quad -\frac{1}{2} x_2 - y_2 \\
\{ & \quad x_2 \in [0, 1] \\
\} & \quad y_2 \in S(x_1, x_2)
\end{align*}
\]

where \( S(x_1, x_2) \) is the solution map of

\[
\min_{y \geq 0} \quad y(-1 + x_1 + x_2) + \frac{1}{2} y^2
\]
Actually $S(x_1, x_2) = \max\{0, 1 - x_1 - x_2\}$ thus the problem becomes

$$
\begin{align*}
\min_{x_1, y_1} & \quad \frac{1}{2} x_1 + y_1 \\
& \begin{cases} 
  x_1 \in [0, 1] \\
  y_1 = \max\{0, 1 - x_1 - x_2\}
\end{cases}
\end{align*}

\begin{align*}
\min_{x_2, y_2} & \quad -\frac{1}{2} x_2 - y_2 \\
& \begin{cases} 
  x_2 \in [0, 1] \\
  y_2 = \max\{0, 1 - x_1 - x_2\}
\end{cases}
\end{align*}
$$
Actually $S(x_1, x_2) = \max\{0, 1 - x_1 - x_2\}$ thus the problem becomes

$$\min_{x_1, y_1} \frac{1}{2} x_1 + y_1 \quad \min_{x_2, y_2} -\frac{1}{2} x_2 - y_2$$

\[
\begin{align*}
\quad \{ x_1 \in [0, 1] \\
y_1 = \max\{0, 1 - x_1 - x_2\}\}
\end{align*}
\[
\begin{align*}
\quad \{ x_2 \in [0, 1] \\
y_2 = \max\{0, 1 - x_1 - x_2\}\}
\end{align*}
\]

Then the Response maps are

$$\mathcal{R}_1(x_2) = \{1 - x_2\} \quad \text{and} \quad \mathcal{R}_2(x_1) = \begin{cases} 
\{0\} & x_1 \in [0, \frac{1}{2}] \\
\{0, 1\} & x_1 = \frac{1}{2} \\
\{1\} & x_1 \in ]\frac{1}{2}, 1]\n\end{cases}$$

and thus there is no Nash equilibrium......
But let us consider the slightly modified problem......

\[ \min_{x_1, y_1} \frac{1}{2} x_1 + y_1 \]
\[ \begin{align*}
    x_1 & \in [0, 1] \\
    y_1 & = \max\{0, 1 - x_1 - x_2\} \\
    y_2 & = \max\{0, 1 - x_1 - x_2\}
\end{align*} \]

\[ \min_{x_2, y_2} -\frac{1}{2} x_2 - y_2 \]
\[ \begin{align*}
    x_2 & \in [0, 1] \\
    y_1 & = \max\{0, 1 - x_1 - x_2\} \\
    y_2 & = \max\{0, 1 - x_1 - x_2\}
\end{align*} \]
But let us consider the slightly modified problem......

\[
\begin{align*}
\min_{x_1, y_1} & \quad \frac{1}{2} x_1 + y_1 \\
& \quad \left\{ \begin{array}{l}
x_1 \in [0, 1] \\
y_1 = \max\{0, 1 - x_1 - x_2\} \\
y_2 = \max\{0, 1 - x_1 - x_2\} 
\end{array} \right. \\
\min_{x_2, y_2} & \quad -\frac{1}{2} x_2 - y_2 \\
& \quad \left\{ \begin{array}{l}
x_2 \in [0, 1] \\
y_1 = \max\{0, 1 - x_1 - x_2\} \\
y_2 = \max\{0, 1 - x_1 - x_2\} 
\end{array} \right.
\end{align*}
\]

that can be proved to have a (unique) Nash equilibrium namely
\((x_1, x_2) = (0, 1)\) with \(y_1 = y_2 = 0!!!!\)

They proved that every Nash equilibrium (initial problem) is a Nash equilibrium for the “all equilibrium” formulation.

It corresponds to the case where each leader takes into account the conjectures regarding the follower decision made by all other leaders....
Some motivation examples
Electricity markets
A short introduction to electricity markets

Didier Aussel
Volume of exchanges

- 20% of exchanges: Spot market
- 80% of exchanges: Long-term contracts
  Mid-term contracts
A short introduction to electricity markets (cont.)

Volume of exchanges

Bid schedule of the spot market

Producers and retailers enter their bids

Producers plan their production = Unit Commitments

Real production/purchases

12:00 12:40

Regulator computes market price and productions/purchases

Didier Aussel
Modeling an Electricity Markets

- electricity market consists of
  i) generators/consumers $i \in \mathcal{N}$ respect their own interests in competition with others
  ii) market operator (ISO) who maintain energy generation and load balance, and protect public welfare

- the ISO has to consider:
  ii) quantities $q_i$ of generated/consumed electricity
  iii) electricity dispatch $t_e$ with respect to transmission capacities
electricity market consists of

i) generators/consumers \( i \in \mathcal{N} \) respect their own interests in competition with others

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the ISO has to consider:

i) quantities \( q_i \) of generated/consumed electricity

ii) electricity dispatch \( t_e \) with respect to transmission capacities

since 1990s, Generalized Nash equilibrium problem is the most popular way of modeling spot electricity markets or, more precisely, Multi-leader-common-follower game
Multi-Leader-Common-Follower game
A classical problem (of a producer) is the best response search.
Models with linear unit bid functions

- **Electricity markets without transmission losses:**
Models with linear unit bid functions

Electricity markets without transmission losses:

Electricity markets with transmission losses:
Some references on the topic (cont.)

- **Best response in electricity markets:**
Some references on the topic (cont.)

- **Best response in electricity markets:**

- **Explicit formula for equilibria**
Non a priori structured bid functions


Let consider a fixed time instant and denote

- $D > 0$ be the overall energy demand of all consumers
- $\mathcal{N}$ be the set of producers
- $q_i \geq 0$ be the production of $i$-th producer, $i \in \mathcal{N}$
Let consider a fixed time instant and denote

- $D > 0$ be the overall energy demand of all consumers
- $\mathcal{N}$ be the set of producers
- $q_i \geq 0$ be the production of $i$-th producer, $i \in \mathcal{N}$

We assume that producer $i \in \mathcal{N}$ provides to the ISO a quadratic bid function $a_i q_i + b_i q_i^2$ given by $a_i, b_i \geq 0$. 

$D > 0$ be the overall energy demand of all consumers

$\mathcal{N}$ be the set of producers

$q_i \geq 0$ be the production of $i$-th producer, $i \in \mathcal{N}$

$a_i, b_i \geq 0$. 

Let consider a fixed time instant and denote

- $D > 0$ be the overall energy demand of all consumers
- $\mathcal{N}$ be the set of producers
- $q_i \geq 0$ be the production of $i$-th producer, $i \in \mathcal{N}$

We assume that producer $i \in \mathcal{N}$ provides to the ISO a quadratic bid function $a_i q_i + b_i q_i^2$ given by $a_i, b_i \geq 0$.

Similarly, let $A_i q_i + B_i q_i^2$ be the true production cost of $i$-th producer with $A_i \geq 0$ and $B_i > 0$ reflecting the increasing marginal cost of production.
Peculiarity of electricity markets is their bi-level structure:

\[
P_i(a_{-i}, b_{-i}, D) = \max_{a_i, b_i} \max_{q_i} a_i q_i + b_i q_i^2 - (A_i q_i + B_i q_i^2)
\]

such that

\[
\begin{cases}
    a_i, b_i \geq 0 \\
    (q_j)_{j \in \mathcal{N}} \in Q(a, b)
\end{cases}
\]

where set-valued mapping \( Q(a, b) \) denotes solution set of

\[
ISO(a, b, D) = \arg\min_q \sum_{i \in \mathcal{N}} (a_i q_i + b_i q_i^2)
\]

such that

\[
\begin{cases}
    q_i \geq 0, \quad \forall i \in \mathcal{N} \\
    \sum_{i \in \mathcal{N}} q_i = D
\end{cases}
\]
Some motivation examples
Industrial Eco-Parks
Example of water management

- In a geographical area, there are different companies $1, \ldots, n$.
- Each of them is buying fresh water (high price) for their production processes.
- Each company generates some "dirty water" and have to pay for discharge.

Stand alone situation
How does it work?

The aims in designing Industrial Eco-park (IEP) are:

a) Reduce cost of production of each company
b) Reduce the environmental impact of the whole production

Thus "Eco" of IEP is at the same time *Economical* and *ecological*.
What is an « Eco-park »?

Example of water management

How to reach these aims?

a) create a network (water tubes) between the companies

b) Eventually install some regeneration unit (cleaning of the water)

It is important to understand that this approach is not limited to water. It can be applied to vapor, gas, coaling fluids, human resources...
An symbolic example of Industrial eco-park is Kalundborg (Danemark)
What is an « Eco-park » ?

In order to convince companies to participate to the Ecopark, our model should guarantee that:

a) **each company** will have a lower cost of production in Eco-park organization than in stand-alone organization

b) the eco-park organization must generate a **lower freshwater consumption** than with a stand-alone organization
The Eco-park design was done through Multi-objective Optimization by the evaluation of Pareto fronts (Gold programming algorithms, scalarization...).
## Stand-alone structure

<table>
<thead>
<tr>
<th>Enterprise</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Water flowrate (tonne/hr)</strong></td>
<td>Fresh</td>
<td>98.33</td>
<td>54.64</td>
<td>186.67</td>
</tr>
<tr>
<td><strong>Cost (MMUSD/year)</strong></td>
<td>Freshwater+discharge</td>
<td>0.28</td>
<td>0.15</td>
<td>0.52</td>
</tr>
<tr>
<td></td>
<td>Reused water</td>
<td>0.01</td>
<td>0.01</td>
<td>0.02</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td>0.28</td>
<td>0.16</td>
<td>0.54</td>
</tr>
</tbody>
</table>

## Eco-park structure: MOO approach

<table>
<thead>
<tr>
<th>Enterprise</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Water flowrate (tonne/hr)</strong></td>
<td>Fresh</td>
<td>88.33</td>
<td>20.00</td>
<td>206.02</td>
</tr>
<tr>
<td></td>
<td>Shared</td>
<td>76.67</td>
<td>61.04</td>
<td>82.00</td>
</tr>
<tr>
<td><strong>Cost (MMUSD/year)</strong></td>
<td>Freshwater+Discharge</td>
<td>0.18</td>
<td>0.11</td>
<td>0.59</td>
</tr>
<tr>
<td></td>
<td>Reused water</td>
<td>0.01</td>
<td>0.02</td>
<td>0.02</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td>0.20</td>
<td>0.13</td>
<td>0.61</td>
</tr>
</tbody>
</table>
The needed change:

...to have an independant designer/regulator

...to have fair solutions for the companies

Thus we propose to use two different possible models:

- Hierarchical optimisation (bi-level optim.)
- Nash game concept between the companies
Single-Leader-Multi-Follower game

**LEADER:** EIP Authority

Minimize Freshwater Consumption

Subject to:
- Objectives of individual plants
- Freshwater unit cost
- Wastewater treatment unit cost
- Subsidy rate fraction

**FOLLOWERS**

- **Plant 1**
  - Minimize Cost$_1$
  - Subject to:
    - Water balances
    - Water quality constraints
    - Topological constraints

- **Plant 2**
  - Minimize Cost$_2$
  - Subject to:
    - Water balances
    - Water quality constraints
    - Topological constraints

- **Plant n**
  - Minimize Cost$_n$
  - Subject to:
    - Water balances
    - Water quality constraints
    - Topological constraints

...
This very difficult problem is treated as follows:

- first we replace the lower-level (convex) optimization problem by their KKT systems; the resulting problem is an Mathematical Programming with Complementarity Constraints (MPCC);
- second the MPCC problem is solved by penalization methods

Numerical results have been obtained with Julia meta-solver coupled with Gurobi, IPOPT and Baron.
## Stand-alone implementation with regeneration units

<table>
<thead>
<tr>
<th>Enterprise</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>Water flowrate (tonne/hr)</em></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fresh</td>
<td>98.33</td>
<td>22.00</td>
<td>97.50</td>
<td>217.83</td>
</tr>
<tr>
<td>Regenerated</td>
<td>0.00</td>
<td>38.17</td>
<td>111.46</td>
<td>149.63</td>
</tr>
<tr>
<td><em>Cost (MMUSD/year)</em></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Freshwater + discharge</td>
<td>0.28</td>
<td>0.06</td>
<td>0.27</td>
<td>0.61</td>
</tr>
<tr>
<td>Reused water</td>
<td>0.01</td>
<td>0.02</td>
<td>0.05</td>
<td>0.08</td>
</tr>
<tr>
<td>Regenerated water</td>
<td>0.00</td>
<td>0.08</td>
<td>0.10</td>
<td>0.27</td>
</tr>
<tr>
<td>Total</td>
<td>0.28</td>
<td>0.17</td>
<td>0.51</td>
<td>0.96</td>
</tr>
</tbody>
</table>

## Nash equilibrium (SLMFG) with regeneration units

<table>
<thead>
<tr>
<th>Enterprise</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td><em>Water flowrate (tonne/hr)</em></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Freshwater + discharge</td>
<td>20.00</td>
<td>20.00</td>
<td>20.00</td>
<td>60.00</td>
</tr>
<tr>
<td>Shared</td>
<td>126.49</td>
<td>149.54</td>
<td>226.66</td>
<td>502.69</td>
</tr>
<tr>
<td>Regenerated</td>
<td>100.62</td>
<td>64.67</td>
<td>166.64</td>
<td>331.93</td>
</tr>
<tr>
<td><em>Cost (MMUSD/year)</em></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Freshwater + discharge</td>
<td>0.04</td>
<td>0.02</td>
<td>0.11</td>
<td>0.17</td>
</tr>
<tr>
<td>Reused water</td>
<td>0.04</td>
<td>0.03</td>
<td>0.08</td>
<td>0.15</td>
</tr>
<tr>
<td>Regenerated water</td>
<td>0.12</td>
<td>0.08</td>
<td>0.19</td>
<td>0.39</td>
</tr>
</tbody>
</table>
More on Single-Leader-Multi-Follower games

D.A & A. Svensson (J. Optim. Theory Appl. 182 (2019))
Existence for optimistic SLMF games

Theorem

Assume that $F$ is lower semi-continuous, and for each follower $i = 1, ..., M$ the objective $f_i$ is continuous and

$$(x, y_i) \mapsto C_i(x, y_i) := \{y_i \mid g_i(x, y) \leq 0\}$$

is a lower semi-continuous set-valued map which has nonempty compact graph. If the graph of GNEP is nonempty, then the SLMF game admits an optimistic solution.
Example of linear pessimistic SLMF game with no solutions

Let us consider the SLMFG with two followers

\[
\min_{x \in [0,4]} \max_{y \in GNEP(x)} -x + (y_1 + y_2).
\]

with

\[
\begin{align*}
\min_{y_1} & \quad y_1 \\
\text{s.t.} & \quad \begin{cases} 
    y_1 \geq 0 \\
    2y_2 - y_1 \leq 2 \\
    y_1 + y_2 \geq x
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\min_{y_2} & \quad y_2 \\
\text{s.t.} & \quad \begin{cases} 
    y_2 \geq 0 \\
    2y_1 - y_2 \leq 2 \\
    y_1 + y_2 \geq x
\end{cases}
\end{align*}
\]
The solution of the parametric GNEP of the followers is given by

\[
\text{GNEP}(x) = \begin{cases} 
(0, 0), (2, 2) & \text{if } x \geq 4, \\
(0, 0) & \text{if } x \in [0, 4[ \\
\emptyset & \text{otherwise.}
\end{cases}
\] (5)

Notice that the function

\[
\varphi_{\text{max}}(x) := \max_{y \in \text{GNEP}(x)} -x + (y_1 + y_2)
\]

\[
= \begin{cases} 
0 & \text{if } x = 4 \\
-x & \text{if } x \in [0, 4[
\end{cases}
\]

is not lower semi-continuous, so that Weierstrass theorem argument cannot be applied. And in fact, the value of the problem of the leader is \(-4\), while there does not exist a point \(x \in [0, 4]\) with that value. The pessimistic linear single-leader-two-follower problem has no optimal solution.
Going back to applications: IEP
Another approach: the blind/control input models

In two very recent works we suggested some reformulations of the optimal design problem:

- under some hypothesis (unique process for each company, linearization in the case of regeneration units), we shown that the optimal design problem can be reformulated as a classical Mixed Integer Linear Programming problem (MILP);
- this problem can be treated with classical tools (CPLEX);

Moreover we inserted a "minimal gain" condition

\[ \text{Cost}_i(x_i, x_{-i}^P, x^R, E) \leq \alpha_i \cdot STC_i, \quad \forall i \in I_P. \]

ensuring that each participating company will gain at minimum \( \alpha \)% on its production cost.
Another approach: the blind/control input models

Theorem

For $E \in \mathcal{E}$ and $x^R \geq 0$ fixed, the equilibrium set $\text{Eq}(x^R, E)$ is given by

$$\text{Eq}(x^R, E) = \left\{ x^P : \forall i \in I_P, \begin{array}{l}
z_i(x_i) + \sum_{(k,i) \in E} x_{k,i} = \sum_{(i,j) \in E} x_{i,j} \\
g_i(x_i) \leq 0 \\
z_i(x_i) \geq 0 \\
\left. x_i \right|_{E_{i,\text{act}}} = 0 \\
x_i \geq 0
\end{array} \right\} \tag{6}$$

Thus, the optimal design problem is equivalent to

$$\min_{E \in \mathcal{E}, x \in \mathbb{R}^{\max(|E|)}} Z(x)$$

$$\begin{array}{l}
x \in X, \\
z_i(x_i) + \sum_{(k,i) \in E} x_{k,i} = \sum_{(i,j) \in E} x_{i,j}, \quad \forall i \in I \\
\left. x_i \right|_{E_{i,\text{act}}} = 0, \quad \forall i \in I \\
g_i(x_i) \leq 0, \quad \forall i \in I \\
z_i(x_i) \geq 0, \quad \forall i \in I \\
\text{Cost}_i(x_i, x^P_i, x^R, E) \leq \alpha_i \cdot \text{STC}_i, \quad \forall i \in I_P \\
x \geq 0.
\end{array} \tag{7}$$
Some results

Figure: The configuration in the case without regeneration units, $\alpha_i = 0.95$ and Coef = 1. Gray nodes are consuming strictly positive fresh water.
Figure: The number of enterprises operating stand-alone and the global freshwater consumption with Coef = 1.
Figure: The number of enterprises operating stand-alone and the global freshwater consumption with $\alpha = 0.99$. 
Some references...

Some references...


MLFG in the setting of quasiconvex optimization

Didier Aussel
Univ. de Perpignan, France

ALOP autumn school - October 14th, 2020

Coauthors: N. Hadjisavvas (Greece and Saudia), M. Pistek (Czech Republic), Jane Ye (Canada)
I - Introduction to quasiconvex optimization
Quasiconvex optimization

Now the case of GNEP...

Quasiconvexity

A function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be quasiconvex on $K$ if,

for all $x, y \in K$ and all $t \in [0, 1]$,

$$f(tx + (1-t)y) \leq \max\{f(x), f(y)\}.$$
A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be *quasiconvex* on $K$ if,

for all $x, y \in K$ and all $t \in [0, 1],$

$$f(tx + (1 - t)y) \leq \max\{f(x), f(y)\}.$$ 

or

for all $\lambda \in \mathbb{R}$, the sublevel set

$$S_\lambda = \{x \in X : f(x) \leq \lambda\}$$

is convex.
A function \( f : X \rightarrow \mathbb{R} \cup \{+\infty\} \) is said to be \textit{quasiconvex} on \( K \) if,

\[
\text{for all } x, y \in K \text{ and all } t \in [0, 1], \quad f(tx + (1 - t)y) \leq \max\{f(x), f(y)\}.
\]

or

\[
\text{for all } \lambda \in \mathbb{R}, \text{ the sublevel set} \quad S_\lambda = \{x \in X : f(x) \leq \lambda\} \text{ is convex.}
\]

or

\( f \) differentiable

\[
f \text{ is quasiconvex} \iff df \text{ is quasimonotone}
\]
A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be **quasiconvex** on $K$ if,

for all $x, y \in K$ and all $t \in [0, 1]$,

$$f(tx + (1 - t)y) \leq \max\{f(x), f(y)\}.$$  

or

for all $\lambda \in \mathbb{R}$, the sublevel set

$$S_\lambda = \{x \in X : f(x) \leq \lambda\}$$ is convex.

or

$f$ differentiable

$$f \text{ is quasiconvex} \iff df \text{ is quasimonotone}$$

or

$$f \text{ is quasiconvex} \iff \partial f \text{ is quasimonotone}$$
Quasiconvexity

- A function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be \textit{quasiconvex} on $K$ if, for all $\lambda \in \mathbb{R}$, the sublevel set

\[ S_\lambda = \{ x \in X : f(x) \leq \lambda \} \text{ is convex.} \]

- A function $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ is said to be \textit{semistrictly quasiconvex} on $K$ if, $f$ is quasiconvex and for any $x, y \in K$,

\[ f(x) < f(y) \Rightarrow f(z) < f(y), \quad \forall z \in [x, y]. \]
Why not a subdifferential for quasiconvex programming?
Why not a subdifferential for quasiconvex programming?

- No (upper) semicontinuity of $\partial f$ if $f$ is not supposed to be Lipschitz
Why not a subdifferential for quasiconvex programming?

- No (upper) semicontinuity of $\partial f$ if $f$ is not supposed to be Lipschitz
- No sufficient optimality condition

$$\bar{x} \in S_{str}(\partial f, C) \iff \bar{x} \in \arg \min_{C} f$$
II - Normal approach

a- First definitions
A first approach

Sublevel set:

\[ S_\lambda = \{ x \in X : f(x) \leq \lambda \} \]
\[ S_\lambda^> = \{ x \in X : f(x) < \lambda \} \]

Normal operator:

Define \( N_f(x) : X \to 2^{X^*} \) by

\[ N_f(x) = N(S_{f(x)}, x) = \{ x^* \in X^* : \langle x^*, y - x \rangle \leq 0, \ \forall y \in S_{f(x)} \}. \]

With the corresponding definition for \( N_f^>(x) \)
But ...

- $N_f(x) = N(S_f(x), x)$ has no upper-semicontinuity properties
- $N_f^>(x) = N(S_f^>(x), x)$ has no quasimonotonicity properties

Example

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(a, b) = \begin{cases} 
|a| + |b|, & \text{if } |a| + |b| \leq 1 \\
|a| + |b|, & \text{if } |a| + |b| > 1 
\end{cases}$$

Then $f$ is quasiconvex.

Consider $x = (10, 0)$, $x^* = (1, 2)$, $y = (0, 10)$ and $y^* = (2, 1)$. We see that $x^* \in N(S_f(x), x)$ and $y^* \in N(S_f^>(x), x)$ (since $|a| + |b| < 1$ implies $(1, 2) \cdot (a - 10, b) \leq 0$ and $(2, 1) \cdot (a, b - 10) \leq 0$) while $\langle x^*, y - x \rangle > 0$ and $\langle y^*, y - x \rangle < 0$. Hence $N_f^>$ is not quasimonotone.
But ...

- $N_f(x) = N(S_f(x), x)$ has no upper-semicontinuity properties
- $N_f^>(x) = N(S_f^>(x), x)$ has no quasimonotonicity properties

**Example**

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(a, b) = \begin{cases} |a| + |b|, & \text{if } |a| + |b| \leq 1 \\ 1, & \text{if } |a| + |b| > 1 \end{cases}.$$ 

Then $f$ is quasiconvex.
Quasiconvex optimization
Now the case of GNEP...

But ...

- $N_f(x) = N(S_f(x), x)$ has no upper-semicontinuity properties
- $N_f^>(x) = N(S_f^>(x), x)$ has no quasimonotonicity properties

Example
Define $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(a, b) = \begin{cases} |a| + |b|, & \text{if } |a| + |b| \leq 1 \\ 1, & \text{if } |a| + |b| > 1 \end{cases}.$$ 

Then $f$ is quasiconvex.
Consider $x = (10, 0)$, $x^* = (1, 2)$, $y = (0, 10)$ and $y^* = (2, 1)$.

We see that $x^* \in N^<(x)$ and $y^* \in N^<(y)$ (since $|a| + |b| < 1$ implies $(1, 2) \cdot (a - 10, b) \leq 0$ and $(2, 1) \cdot (a, b - 10) \leq 0$) while $\langle x^*, y - x \rangle > 0$ and $\langle y^*, y - x \rangle < 0$. Hence $N^<$ is not quasimonotone.
But ...another example

- $N_f(x) = N(S_f(x), x)$ has no upper-semicontinuity properties
- $N^>_f(x) = N(S^>_f(x), x)$ has no quasimonotonicity properties

Example

Then $f$ is quasiconvex.
But ...another example

- \( N_f(x) = N(S_f(x), x) \) has no upper-semicontinuity properties
- \( N_{f^>}(x) = N(S_{f^>}(x), x) \) has no quasimonotonicity properties

Example

Then \( f \) is quasiconvex.

We easily see that \( N(x) \) is not upper semicontinuous....
But ...another example

- \( N_f(x) = N(S_f(x), x) \) has no upper-semicontinuity properties
- \( N_f^>(x) = N(S_f^>(x), x) \) has no quasimonotonicity properties

Example

Then \( f \) is quasiconvex.

We easily see that \( N(x) \) is not upper semicontinuous....

These two operators are essentially adapted to the class of semi-strictly quasiconvex functions. Indeed in this case, for each \( x \in \text{dom } f \setminus \text{arg min } f \), there is \( S_f^>(x) \) and \( S_f^<(x) \) whose convex envelopes are equal to \( N_f(x) = N_f^>(x) \).
II - Normal approach

b- Adjusted sublevel sets and normal operator
Adjusted sublevel set

For any $x \in \text{dom } f$, we define

$$S^a_f(x) = S_f(x) \cap \overline{B}(S^<_f(x), \rho_x)$$

where $\rho_x = \text{dist}(x, S^<_f(x))$, if $S^<_f(x) \neq \emptyset$

and $S^a_f(x) = S_f(x)$ if $S^<_f(x) = \emptyset$. 

MLFG in the setting of quasiconvex optimization
Adjusted sublevel set

For any \( x \in \text{dom } f \), we define

\[
S^a_f(x) = S_f(x) \cap \overline{B}(S_f^<(x), \rho_x)
\]

where \( \rho_x = \text{dist}(x, S_f^<(x)) \), if \( S_f^<(x) \neq \emptyset \)

and \( S^a_f(x) = S_f(x) \) if \( S_f^<(x) = \emptyset \).

\( S^a_f(x) \) coincides with \( S_f(x) \) if \( \text{cl}(S_f^>(x)) = S_f(x) \)
**Definition**

**Adjusted sublevel set**

For any $x \in \text{dom } f$, we define

$$S^a_f(x) = S_f(x) \cap \overline{B}(S^<_f(x), \rho_x)$$

where $\rho_x = \text{dist}(x, S^<_f(x))$, if $S^<_f(x) \neq \emptyset$

and $S^a_f(x) = S_f(x)$ if $S^<_f(x) = \emptyset$.

- $S^a_f(x)$ coincides with $S_f(x)$ if $\text{cl}(S^>_f(x)) = S_f(x)$
  
  e.g. $f$ is semistrictly quasiconvex
**Definition**

**Adjusted sublevel set**

For any $x \in \text{dom } f$, we define

$$S^a_f(x) = S_f(x) \cap \overline{B}(S^<_f(x), \rho_x)$$

where $\rho_x = \text{dist}(x, S^<_f(x))$, if $S^<_f(x) \neq \emptyset$

and $S^a_f(x) = S_f(x)$ if $S^<_f(x) = \emptyset$.

- $S^a_f(x)$ coincides with $S_f(x)$ if $\text{cl}(S^>_f(x)) = S_f(x)$
  
e.g. $f$ is semistrictly quasiconvex

**Proposition**

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be any function, with domain $\text{dom } f$. Then

$f$ is quasiconvex $\iff$ $S^a_f(x)$ is convex, $\forall x \in \text{dom } f$. 
Adjusted sublevel set:
For any $x \in \text{dom } f$, we define

$$S^a_f(x) = S_f(x) \cap B(S^<_f(x), \rho_x)$$

where $\rho_x = \text{dist}(x, S^<_f(x))$, if $S^<_f(x) \neq \emptyset$.

Adjusted normal operator:

$$N^a_f(x) = \{x^* \in X^* : \langle x^*, y - x \rangle \leq 0, \ \forall y \in S^a_f(x)\}$$
Example

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MLFG in the setting of quasiconvex optimization
Example

\[ \overline{B}(S_{f(x)}^<, \rho_x) \]

\[ S_f^a(x) = S_f(x) \cap \overline{B}(S_{f(x)}^<, \rho_x) \]
Example

\[ S^a_f(x) = S_f(x) \cap \overline{B}(S^{<}_f(x), \rho_x) \]

\[ N^a_f(x) = \{ x^* \in X^* : \langle x^*, y - x \rangle \leq 0, \quad \forall y \in S^a_f(x) \} \]
An exercice........

Let us draw the normal operator value $N^a_f(x,y)$ at the points $(x, y) = (0.5, 0.5), (x, y) = (0, 1), (x, y) = (1, 0), (x, y) = (1, 2), (x, y) = (1.5, 0)$ and $(x, y) = (0.5, 2)$. 
Let us draw the normal operator value $N^a_i(x, y)$ at the points $(x, y) = (0.5, 0.5)$, $(x, y) = (0, 1)$, $(x, y) = (1, 0)$, $(x, y) = (1, 2)$, $(x, y) = (1.5, 0)$ and $(x, y) = (0.5, 2)$.

Operator $N^a_i$ provide information at any point!!!
Basic properties of $N_f^a$

**Nonemptyness:**

**Proposition**

*Let $f : X \to \mathbb{IR} \cup \{+\infty\}$ be lsc. Assume that rad. continuous on $\text{dom } f$ or $\text{dom } f$ is convex and $\text{int } S_\lambda \neq \emptyset$, $\forall \lambda > \inf_X f$. Then*

$f$ is quasiconvex $\iff N_f^a(x) \setminus \{0\} \neq \emptyset$, $\forall x \in \text{dom } f \setminus \text{arg min } f$.

**Quasimonotonicity:**

The normal operator $N_f^a$ is always quasimonotone
Upper sign-continuity

- \( T : X \to 2^X \) is said to be **upper sign-continuous** on \( K \) iff for any \( x, y \in K \), one have:

\[
\forall t \in ]0, 1[, \quad \inf_{x^* \in T(x_t)} \langle x^*, y - x \rangle \geq 0
\]

\[
\implies \sup_{x^* \in T(x)} \langle x^*, y - x \rangle \geq 0
\]

*where \( x_t = (1 - t)x + ty \).*
Definition

Let \( T : K \to 2^{X^*} \) be a set-valued map.

\( T \) is called locally upper sign-continuous on \( K \) if, for any \( x \in K \) there exist a neigh. \( V_x \) of \( x \) and a upper sign-continuous set-valued map \( \Phi_x(\cdot) : V_x \to 2^{X^*} \) with nonempty convex \( w^* \)-compact values such that

\[
\Phi_x(y) \subseteq T(y) \setminus \{0\}, \ \forall \ y \in V_x
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Continuity of normal operator

Proposition

Let $f$ be lsc quasiconvex function such that $\text{int}(S_\lambda) \neq \emptyset$, $\forall \lambda > \inf f$.

Then $N_f^\lambda$ is locally upper sign-continuous on $\text{dom } f \setminus \text{arg min } f$. 
Quasiconvex optimization

Now the case of GNEP...

III

Quasiconvex programming

a- Optimality conditions
Let $f : X \to \mathbb{IR} \cup \{+\infty\}$ and $K \subseteq \text{dom} \ f$ be a convex subset.

$$(P) \quad \text{find } \bar{x} \in K : f(\bar{x}) = \inf_{x \in K} f(x)$$
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**Perfect case: $f$ convex**

$f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ a proper convex function

$K$ a nonempty convex subset of $X$, $\bar{x} \in K + \text{C.Q.}$

Then

$$f(\bar{x}) = \inf_{x \in K} f(x) \iff \bar{x} \in S_{str}(\partial f, K)$$
Quasiconvex programming

Let \( f : X \to \mathbb{R} \cup \{+\infty\} \) and \( K \subseteq \text{dom } f \) be a convex subset.

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\]

**What about \( f \) quasiconvex case?**

\[
\bar{x} \in S_{str}(\partial f(\bar{x}), K) \Rightarrow \bar{x} \in \arg \min_{K} f
\]
Sufficient optimality condition

Theorem

\[ f : X \to \mathbb{IR} \cup \{+\infty\} \text{ quasiconvex, radially cont. on } \text{dom } f \]

\[ C \subseteq X \text{ such that } \text{conv}(C) \subset \text{dom } f. \]

Suppose that \( C \subset \text{int}(\text{dom } f) \) or \( \text{Aff}C = X. \)

Then \( \bar{x} \in S(N^a_f \setminus \{0\}, C) \implies \forall x \in C, f(\bar{x}) \leq f(x). \)

where \( \bar{x} \in S(N^a_f \setminus \{0\}, K) \) means that there exists \( \bar{x}^* \in N^a_f(\bar{x}) \setminus \{0\} \) such that

\[ \langle \bar{x}^*, c - x \rangle \geq 0, \quad \forall c \in C. \]
Proposition

Let $C$ be a closed convex subset of $X$, $\bar{x} \in C$ and $f : X \to \mathbb{R}$ be continuous semistrictly quasiconvex such that $\text{int}(S^a_f(\bar{x})) \neq \emptyset$ and $f(\bar{x}) > \inf_X f$. Then the following assertions are equivalent:

i) $f(\bar{x}) = \min_C f$

ii) $\bar{x} \in S_{str}(N^a_f \setminus \{0\}, C)$

iii) $0 \in N^a_f(\bar{x}) \setminus \{0\} + NK(C, \bar{x})$. 

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MLFG in the setting of quasiconvex optimization
GNEP reformulation in quasiconvex case

To simplify the notations, we will denote, for any $i$ and any $x \in \mathbb{R}^n$, by $S_i(x)$ and $A_i(x^{-i})$ the subsets of $\mathbb{R}^{n_i}$

$$S_i(x) = S_{\theta_i(\cdot, x^{-i})}^a(x^i) \quad \text{and} \quad A_i(x^{-i}) = \arg \min_{\mathbb{R}^{n_i}} \theta_i(\cdot, x^{-i}).$$

In order to construct the variational inequality problem we define the following set-valued map $N_\theta^a : \mathbb{R}^n \to 2^{\mathbb{R}^n}$ which is described,

for any $x = (x^1, \ldots, x^p) \in \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_p}$, by

$$N_\theta^a(x) = F_1(x) \times \ldots \times F_p(x),$$

where $F_i(x) = \begin{cases} \overline{B_i}(0, 1) & \text{if } x^i \in A_i(x^{-i}) \\ \operatorname{co}(N_{\theta_i}^a(x^i) \cap S_i(0, 1)) & \text{otherwise} \end{cases}$

The set-valued map $N_\theta^a$ has nonempty convex compact values.
In the following we assume that $X$ is a given nonempty subset $X$ of $\mathbb{R}^n$, such that for any $i$, the set $X_i(x^{-i})$ is given as

$$X_i(x^{-i}) = \{x^i \in \mathbb{R}^n : (x^i, x^{-i}) \in X\}.$$ 

**Theorem**

Let us assume that, for any $i$, the function $\theta_i$ is continuous and quasiconvex with respect to the $i$-th variable. Then every solution of $S(N^\theta, X)$ is a solution of the GNEP.
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**Theorem**

*Let us assume that, for any $i$, the function $\theta_i$ is continuous and quasiconvex with respect to the $i$-th variable. Then every solution of $S(N_{\theta}, X)$ is a solution of the GNEP.*

Note that the link between GNEP and variational inequality is valid even if the constraint set $X$ is neither convex nor compact.
Lemma

Let $i \in \{1, \ldots, p\}$. If the function $\theta_i$ is continuous quasiconvex with respect to the $i$-th variable, then,

$$0 \in F_i(\bar{x}) \iff \bar{x}^i \in A_i(\bar{x}^{-i}).$$

Proof. It is sufficient to consider the case of a point $\bar{x}$ such that $\bar{x}^i \not\in A_i(\bar{x}^{-i})$. Since $\theta_i(\cdot, \bar{x}^{-i})$ is continuous at $\bar{x}^i$, the interior of $S_i(\bar{x})$ is nonempty. Let us denote by $K_i$ the convex cone

$$K_i = N_{\theta_i}^a(\bar{x}^i) = (S_i(\bar{x}) - \bar{x}^i)^{\circ}.$$ 

By quasiconvexity of $\theta_i$, $K_i$ is not reduced to $\{0\}$. Let us first observe that, since $S_i(\bar{x})$ has a nonempty interior, $K_i$ is a pointed cone, that is $K_i \cap (-K_i) = \{0\}$.

Now let us suppose that $0 \in F_i(\bar{x})$. By Caratheodory theorem, there exist vectors $v_i \in [K_i \cap S_i(0, 1)]$, $i = 1, \ldots, n + 1$ and scalars $\lambda_i \geq 0$, $i = 1, \ldots, n + 1$ with

$$\sum_{i=1}^{n+1} \lambda_i = 1 \text{ and } 0 = \sum_{i=1}^{n+1} \lambda_i v_i.$$
Since there exists at least one \( r \in \{1, \ldots, n+1\} \) such that \( \lambda_r > 0 \) we have

\[
v_r = - \sum_{i=1, i \neq r}^{n+1} \frac{\lambda_i}{\lambda_r} v_i
\]

which clearly shows that \( v_r \) is an element of the convex cone \(-K_i\). But \( v_r \in S_i(0, 1) \) and thus \( v_r \neq 0 \). This contradicts the fact that \( K_i \) is pointed and the proof is complete. \( \blacksquare \)
Proof. Let us consider $\bar{x}$ to be a solution of $S(N^a_{\theta}, X)$. There exists $v \in N^a_{\theta}(\bar{x})$ such that

$$\langle v, y - \bar{x} \rangle \geq 0, \quad \forall y \in X. \quad (*)$$

Let $i \in \{1, \ldots, p\}$.

If $\bar{x}^i \in A_i(\bar{x}^{-i})$ then obviously $\bar{x}^i \in Sol_i(\bar{x}^{-i})$.

Otherwise $v^i \in F_i(\bar{x}) = co(N^a_{\theta_i}(\bar{x}^i) \cap S(0, 1))$. Thus, according to Lemma 2, there exist $\lambda > 0$ and $u^i \in N^a_{\theta_i}(\bar{x}^i) \setminus \{0\}$ satisfying $v^i = \lambda u^i$.

Now for any $x^i \in X_i(\bar{x}^{-i})$, consider $y = (\bar{x}^1, \ldots, \bar{x}^{i-1}, x^i, \bar{x}^{i+1}, \ldots, \bar{x}^p)$.

From $(*)$ one immediately obtains that $\langle u^i, x^i - \bar{x}^i \rangle \geq 0$. Since $x^i$ is an arbitrary element of $X_i(\bar{x}^{-i})$, we have that $\bar{x}^i$ is a solution of $S(N^a_{\theta_i} \setminus \{0\}, X_i(\bar{x}^{-i}))$ and therefore, according to Prop. 4,

$$\bar{x}^i \in Sol_i(\bar{x}^{-i})$$

Since $i$ was arbitrarily chosen we conclude that $\bar{x}$ solves the GNEP.
Let us suppose that, for any $i$, the loss function $\theta_i$ is continuous and semistrictly quasiconvex with respect to the $i$-th variable. Further assume that the set $X$ is a nonempty convex subset of $\mathbb{R}^N$. Then

any solution of the variational inequality $S(N_\theta^a, X)$ is a solution of the GNEP

any solution of the GNEP is a solution of the quasi-variational inequality $QVI(N_\theta^a, \mathcal{X})$

and where $\mathcal{X}$ stands for the set-valued map defined on $\mathbb{R}^2$ by

$$\mathcal{X}(x) = \prod_{i=1}^{p} X_i(x^{-i})$$