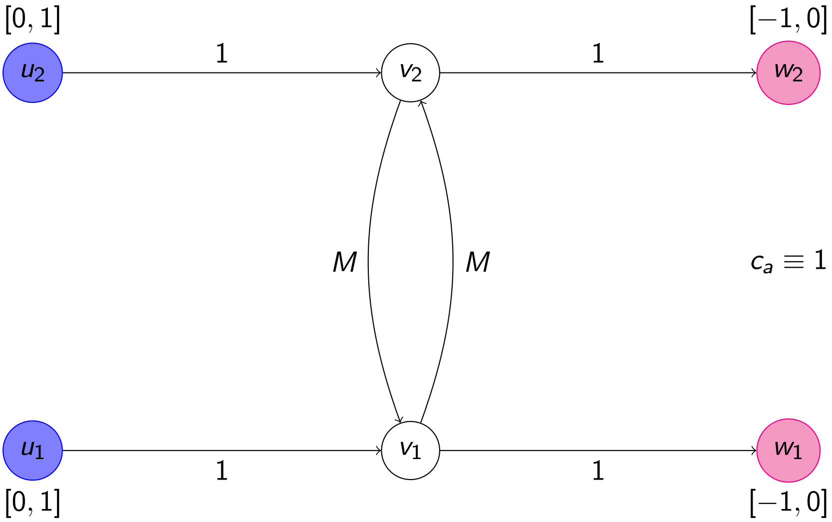
The Maximum Minimum Cost Flow Problem

Kai Hoppmann-Baum

ALOP Autumn School on Bilevel Optimization

Example Flow Network



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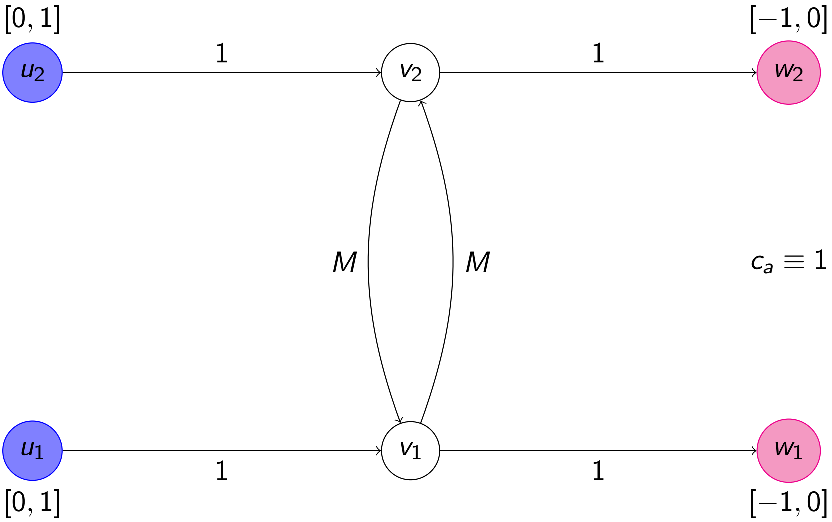
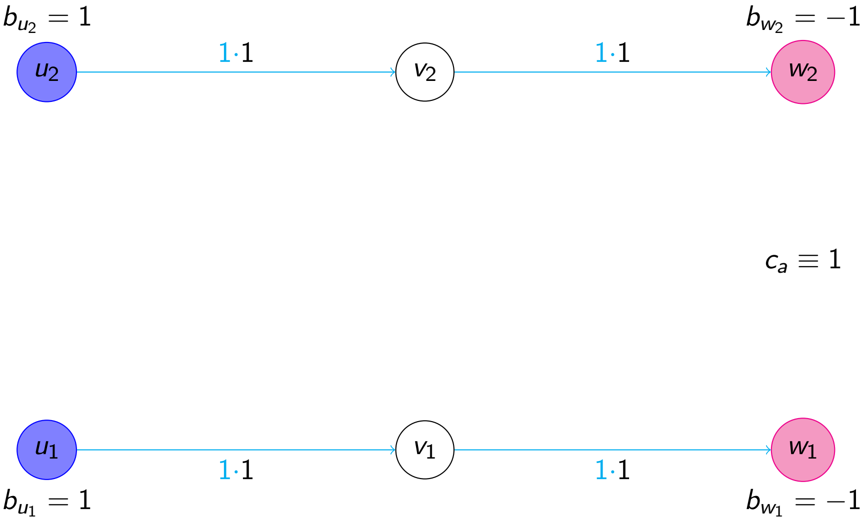


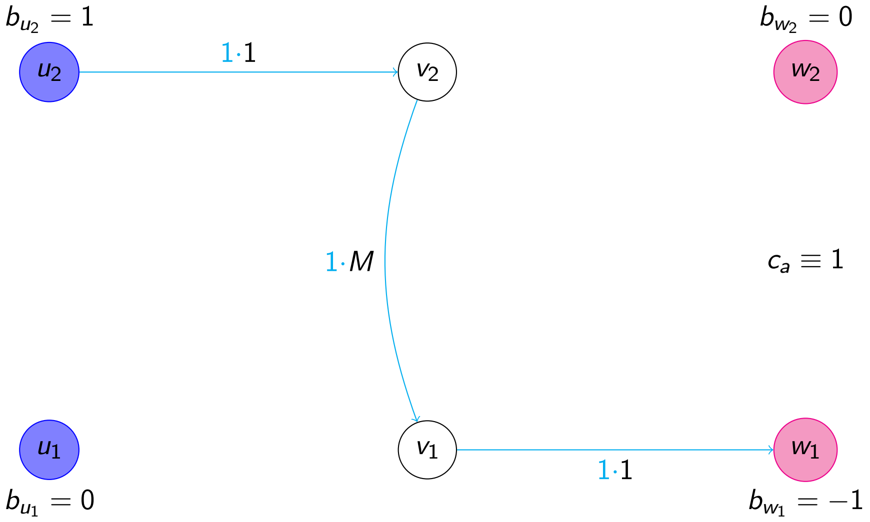
Difficulty ≈ Optimal solution value of induced MCF instance $\sum_{a \in A} \ell_a f_a$.

Greedy Solution - Insert Maximum Possible Flow



Optimal MCF solution value is 4.

Solution with Maximum Difficulty



Optimal MCF solution value is $M + 2$.

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Lemma (Hoppmann '19)

MMCF is NP-hard and remains NP-hard in the uncapacitated case.

Bilevel Programming Model

$$
\begin{aligned}
\max_{b} \quad & \sum_{a \in A} \ell_a f_a \\
\text{s.t.} \quad & b_v \in [\, \underline{b}_v, \overline{b}_v] && \forall v \in V^+ \cup V^- \\
& \min_{f} \quad \sum_{a \in A} \ell_a f_a \\
& \text{s.t.} \quad \sum_{a \in \delta^+(u)} f_a - \sum_{a \in \delta^-(u)} f_a = b_u && \forall u \in V^+ \\
& \qquad \sum_{a \in \delta^+(w)} f_a - \sum_{a \in \delta^-(w)} f_a = b_w && \forall w \in V^- \\
& \qquad \sum_{a \in \delta^+(v)} f_a - \sum_{a \in \delta^-(v)} f_a = 0 && \forall v \in V \setminus (V^+ \cup V^-) \\
& \qquad f_a \in [0, c_a] && \forall a \in A
\end{aligned}
$$

Current Work

- Tighten variable bounds and big-Ms in KKT-reformulation.
- Reduce size of flow network or derive equivalent graph representation with less nodes and/or arcs.
- Derive stronger hardness results on capacitated variant.

The Last Slide

Thank you for your attention!

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