Characterizations of Multiobjective Robustness via Oriented Distance Function

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Uncertain Multiobjective Optimization Problem

- $\emptyset \neq \mathcal{X} \subset X$: real vector space.
- Y : real normed space.
- $\emptyset \neq \mathcal{U}$: compact subset of a finite dimensional space.
- $f : \mathcal{X} \times \mathcal{U} \to Y$, $g_i : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$, $i = 1, \ldots, m$ and $g(x, \xi) := (g_1(x, \xi), \ldots, g_m(x, \xi))$, where $\xi \in \mathcal{U}$ represents an uncertain parameter,

The uncertain multiobjective optimization problem $\mathrm{P}(\mathcal{U})$ is defined by

$$\min f(x, \xi) \text{ such that } g(x, \xi) \in \mathcal{D},\ x \in \mathcal{X},$$

where $\mathcal{D} \subseteq -\mathbb{R}^m_+$ is a proper, closed and convex cone.

- The deterministic robust counterpart to $\mathrm{P}(\mathcal{U})$ given by

$$\min \sup_{\xi \in \mathcal{U}} f(x, \xi), \text{ such that } \forall \xi \in \mathcal{U} : g(x, \xi) \in \mathcal{D}, \, x \in \mathcal{X}. \quad (1)$$

- $\mathcal{R} := \{x \in \mathcal{X} : g(x, \xi) \in \mathcal{D}, \forall \xi \in \mathcal{U}\}$: feasible set.
- $f_{\mathcal{U}}(x) := \{f(x, \xi) : \xi \in \mathcal{U}\} \subseteq Y$.

Moreover, we take

$$\mathcal{G}(x) := \left(\sup_{\xi \in \mathcal{U}} g_1(x, \xi), \ldots, \sup_{\xi \in \mathcal{U}} g_m(x, \xi) \right).$$

- Assume that the sets $f_{\mathcal{U}}(x)$ are compact for all $x \in \mathcal{X}$, and $\sup_{\xi \in \mathcal{U}} g_i(x, \xi) < \infty$, $i = 1, \ldots, m$ for all $x \in \mathcal{X}$.

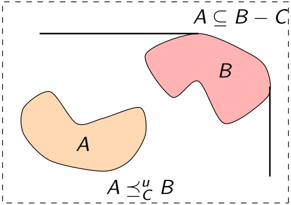
Set Order Relation

Let A, B be nonempty subsets of Y and C be a proper convex cone in Y.

- The upper set less order relation $\preceq_C^u$ is defined by

$$A \preceq_C^u B \quad \Leftrightarrow \quad A \subseteq B - C,$$

or equivalently, for all $a \in A$, there exists $b \in B$ such that $a \preceq_C b$.



- The set order relation $\preceq_C^u$ is pre-order, that is, reflexive and transitive relation. In general, the set order relation $\preceq_C^u$ is not antisymmetric.

Corresponding to set order relation $\preceq_C^u$, we have the following definition of robustness concepts.

Definition 1

Given an uncertain multiobjective optimization problem $P(\mathcal{U})$, a feasible solution $\bar{x} \in \mathcal{R}$ is called $\preceq_C^u$-robust for $P(\mathcal{U})$, if there is no feasible solution $x_0 \in \mathcal{R} \setminus \{\bar{x}\}$ such that $f_{\mathcal{U}}(x_0) \preceq_C^u f_{\mathcal{U}}(\bar{x})$.

Oriented Distance Function

- Let $\emptyset \neq M \subseteq Y$. The function $\triangle_M : Y \to \mathbb{R} \cup \{\pm\infty\} =: \overline{\mathbb{R}}$ defined by

$$\triangle_M (y) := d_M(y) - d_{Y \setminus M}(y), \quad \forall y \in Y \tag{2}$$

  is called oriented distance function, where $d_M(y) := \inf_{m \in M} \|y - m\|$ is the distance function from $y \in Y$ to the set M.

- If $\inf_{b \in B} \triangle_C(b - a)$ is attained for all $a \in A$, then

$$A \preceq_C^u B \quad \Leftrightarrow \quad \sup_{a \in A} \inf_{b \in B} \triangle_C(b - a) \leq 0. \tag{3}$$

Let $\bar{x} \in \mathcal{X}$. Define a map $\mathcal{A}_{\bar{x}} : \mathcal{X} \to \mathbb{R}^{1+m}$ by

$$\mathcal{A}_{\bar{x}}(x) := \left(\sup_{\xi \in \mathcal{U}} \inf_{\eta \in \mathcal{U}} \triangle_C(f(\bar{x}, \eta) - f(x, \xi)), \mathcal{G}(x) \right), \quad \forall x \in \mathcal{X},$$

and consider the set

$$\mathcal{K}_{\bar{x}} := \left\{ (u, v) \in \mathbb{R}^{1+m} \, : \, (u, v) = \mathcal{A}_{\bar{x}}(x), \, x \in \mathcal{X} \setminus \{\bar{x}\} \right\},$$

which is the corrected scalarization image of problem $\mathrm{P}(\mathcal{U})$, and the set

$$\mathcal{H} := \left\{ (u, v) \in \mathbb{R}^{1+m} \, : \, u \leq 0, \, v \in \mathcal{D} \right\},$$

where $\mathbb{R}^{1+m}$ denotes the scalarization image space.

Theorem 2

Let $\bar{x} \in \mathcal{R}$ such that for each $x \in \mathcal{R} \setminus \{\bar{x}\}$, the infimum $\inf_{\eta \in \mathcal{U}} \triangle_C(f(\bar{x}, \eta) - f(x, \xi))$ is attained for all $\xi \in \mathcal{U}$. Then a feasible solution $\bar{x} \in \mathcal{R}$ is $\preceq^u_C$-robust for the problem $P(\mathcal{U})$ if and only if the generalized system $\mathcal{A}_{\bar{x}}(x) \in \mathcal{H}$, $x \in \mathcal{X} \setminus \{\bar{x}\}$ has no solutions, or equivalently, $\mathcal{K}_{\bar{x}} \cap \mathcal{H} = \emptyset$.

Application to Shortest Path Problem

Suppose we want to travel between two specified points A and B. We are interested in a short travel time and in low costs. Unfortunately, both, cost and travel time, are not known beforehand. They both depend on the decision, if some festival event takes place or not. Suppose, we have two possible paths x and y. Let $\mathcal{U} = \{\xi_1, \xi_2\}$ be the uncertainty set with two scenarios, where ξ_1 represents no festival and ξ_2 represents festival.

$f(x,\xi)$	x	y
ξ_1	$(1,2)$	$(5,1.9)$
ξ_2	$(2.5,1.5)$	$(2,5)$

Table 1: The function values $f(x,\xi)$ are summarized for the paths x and y under two scenarios ξ_1 and ξ_2.

- Set $\bar{x} = x$ and $C = \mathbb{R}^2_+$. After a short calculation, we get

$$\inf_{\eta \in \mathcal{U}} \triangle_C(f(x,\eta) - f(y,\xi_1)) > 0,$$

  and

$$\inf_{\eta \in \mathcal{U}} \triangle_C(f(x,\eta) - f(y,\xi_2)) > 0.$$

  Thus, $\sup_{\xi \in \mathcal{U}} \inf_{\eta \in \mathcal{U}} \triangle_C(f(x,\eta) - f(y,\xi)) > 0$. Hence, $\mathcal{K}_x \cap \mathcal{H} = \emptyset$.
- Set $\bar{x} = y$. Again, after a short calculation, we get

$$\inf_{\eta \in \mathcal{U}} \triangle_C(f(y,\eta) - f(x,\xi_1)) < 0,$$

  and

$$\inf_{\eta \in \mathcal{U}} \triangle_C(f(y,\eta) - f(x,\xi_2)) < 0.$$

  Thus, $\sup_{\xi \in \mathcal{U}} \inf_{\eta \in \mathcal{U}} \triangle_C(f(y,\eta) - f(x,\xi)) < 0$. Hence, $\mathcal{K}_y \cap \mathcal{H} \neq \emptyset$.

Using Theorem 2, x is $\preceq^u_C$-upper set less ordered strictly efficient.

The presentation was extracted from the following paper.

[1] Q.H. Ansari, E. Köbis and P.K. Sharma, *Characterizations of multiobjective robustness via oriented distance function and image space analysis.* J. Optim. Theory and Appl. **181**(3), 817–839 (2019).

Questions?

and

Thanks!