Introduction to Bilevel Optimitization,Linear Bilevel Problems, and Maybe Beyond

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Outline

- Bilevel optimisation
- Linear Bilevel problems
- Bilevel problems with bilinear objectives

Bilevel optimization:
what is it and a bit of history

Bilevel Optimization Problem

$$
\begin{aligned}
\max_{x,y} \quad & f(x,y) \\
\text{s.t.} \quad & (x,y) \in X \\
& y \in S(x) \\
\text{where} \quad & S(x) = \operatorname*{argmax}_{y} g(x,y) \\
& \text{s.t.} (x,y) \in Y
\end{aligned}
$$

Adequate framework for Stackelberg game

- Leader: 1st level,
- Follower: 2nd level,
- Leader takes follower's optimal reaction into account.

Heinrich von Stackelberg
(1905 - 1946)

First OR paper on bilevel optimization

Bracken & McGill (Op.Res.1973): First bilevel model, structural properties, military application.

Mathematical Programs with Optimization Problems in the Constraints

Jerome Bracken and James T. McGill

Institute for Defense Analyses, Arlington, Virginia

(Received October 5, 1971)

This paper considers a class of optimization problems characterized by constraints that themselves contain optimization problems. The problems in the constraints can be linear programs, nonlinear programs, or two-sided optimization problems, including certain types of games. The paper presents theory dealing primarily with properties of the relevant functions that result in convex programming problems, and discusses interpretations of this theory. It gives an application with linear programs in the constraints, and discusses computational methods for solving the problems.

Applications

- Economic game theory
- Production planning
- Revenue management
- Security
- …

Bilevel optimization: Some examples

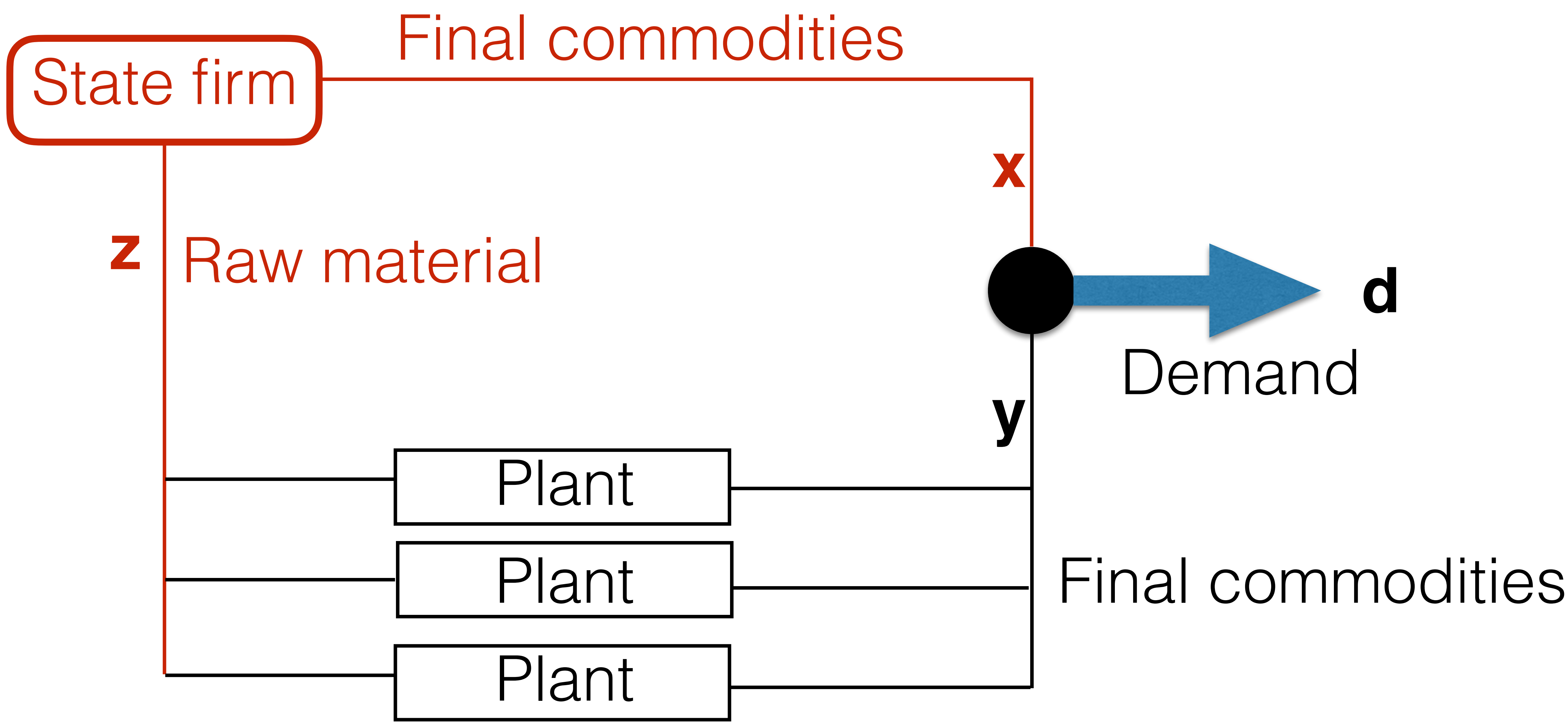
A production planning problem (Maravillo, Camacho-Vallejo, Puerto, Labbé 2020)

Two vertically integrated industries:

- First industry: state-owned firm, monopole for the production of raw materials (supplies) for the second industry.
- Second industry: private firm + state-owned firm compete to produce commodities.

- All firms have a limited production capacity.
- The state-owned firm must achieve a minimum income.

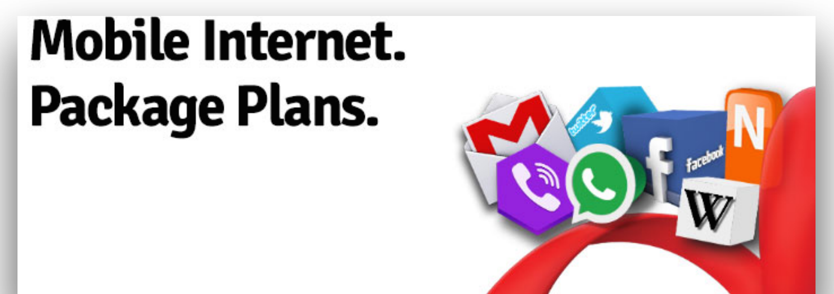
Problem description



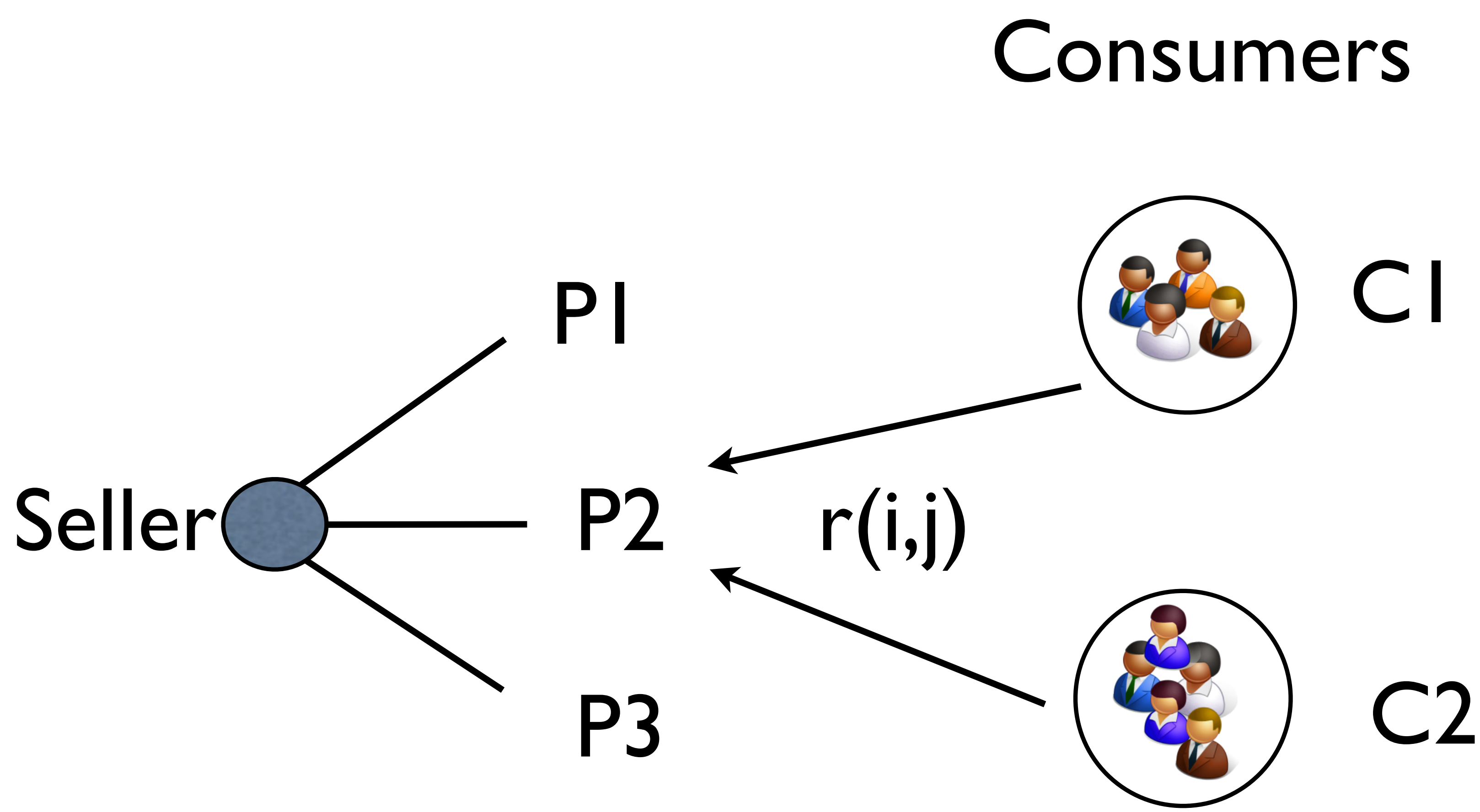
Bilevel formulation

$$
\begin{aligned}
&\min_{x,z,r,s,y} \sum_{i\in I}(r_i + s_i) \\
&\text{subject to: } \frac{\sum_{j\in J} y_{ij} + x_i}{d_i} + r_i - s_i = 1 && \forall i \\
&t \leq \sum_{i\in I}(p_i - c_i^G)x_i \\
&0 \leq x_i \leq q_i^A && \forall i \\
&0 \leq z_i \leq q_i^B && \forall i \\
&r_i \geq 0 && \forall i \\
&s_i \geq 0 && \forall i \\
&y \in \operatorname{argmax} \sum_{j\in J}\sum_{i\in I}(p_i - c_{ij}^E)\hat{y}_{ij} \\
&\qquad \text{subject to: } \sum_{j\in J} a_{ij}\hat{y}_{ij} \leq z_i && \forall i \\
&\qquad \sum_{i\in I} b_{ij}\hat{y}_{ij} \leq m_j && \forall j \\
&\qquad \hat{y}_{ij} \geq 0 && \forall i, \forall j
\end{aligned}
$$

Applications in revenue management



Product pricing problem (Dobson and Kalish 1988)



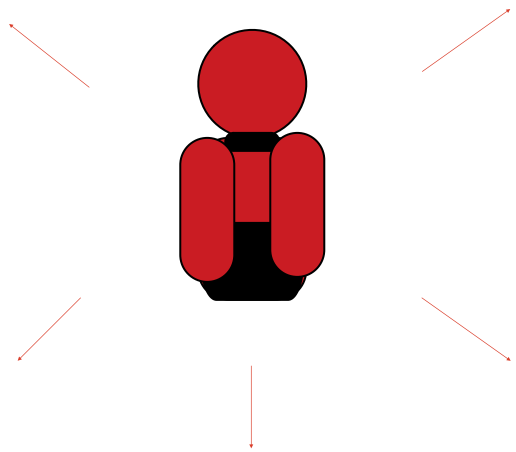
- p_i = price of product i
- $r(i, j)$ = reservation price of consumer group C_j for product i

Product pricing problem

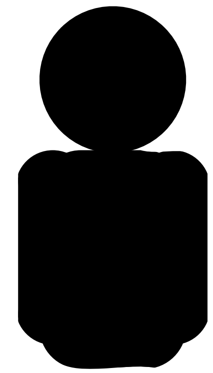
$$
\begin{aligned}
\max_{p} \quad & \sum_i \sum_j p_i x_{ij} \\
\text{s.t.} \quad & \max_{x} \sum_i (r(i,j) - p_i) x_{ij} \\
& \text{s.t.} \quad \sum_i x_{ij} \leq 1 \\
& \qquad\quad x_{ij} \geq 0, \forall i, j
\end{aligned}
$$

Applications in control, protection and security

(Tambe et al., USC)



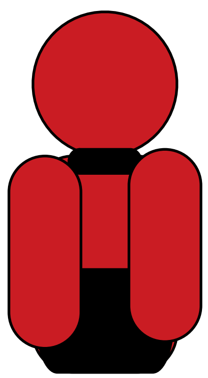
Stackelberg Bimatrix game



Follower

	C	D	
A	(2,1)	(4,0)	0.5
B	(1,0)	(3,2)	0.5

Leader Mixed Strategy



Leader

Stackelberg solution:

Pure strategies: (B,D) with payoff of (3,2)
Mixed strategies: (A:1/2, B:1/2; D) with a payoff of (3.5,1)

Stackelberg vs Nash
(pure strategies)

	Player 2 - A	**Player 2 - B**
Player 1 - A	***(2,2)***	***(4,1)***
Player 1 - B	(1,0)	(3,1)

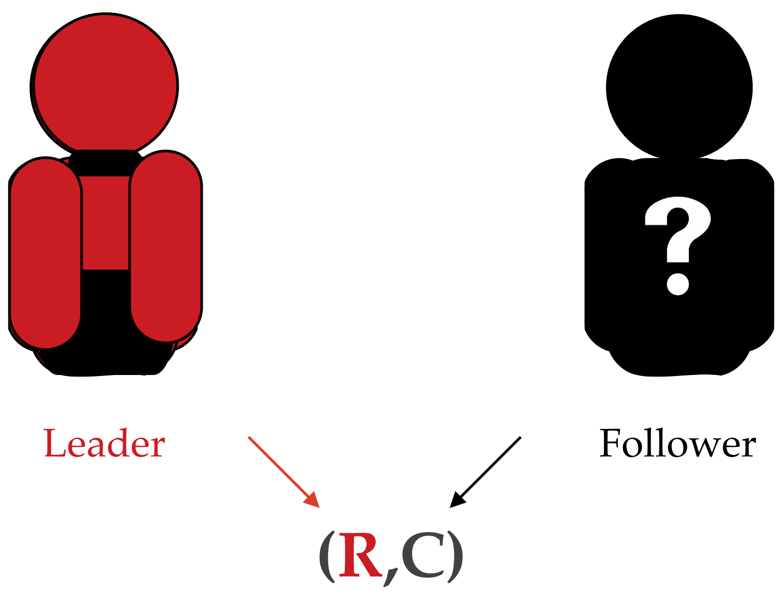
Nash equilibrium: Player 1-A and Player 2-A => (2,2)

Stackelberg solution: Player 1-B and Player 2-B => (3,1)

Nash equilibrium may not exist
There is always a Stackelberg solution

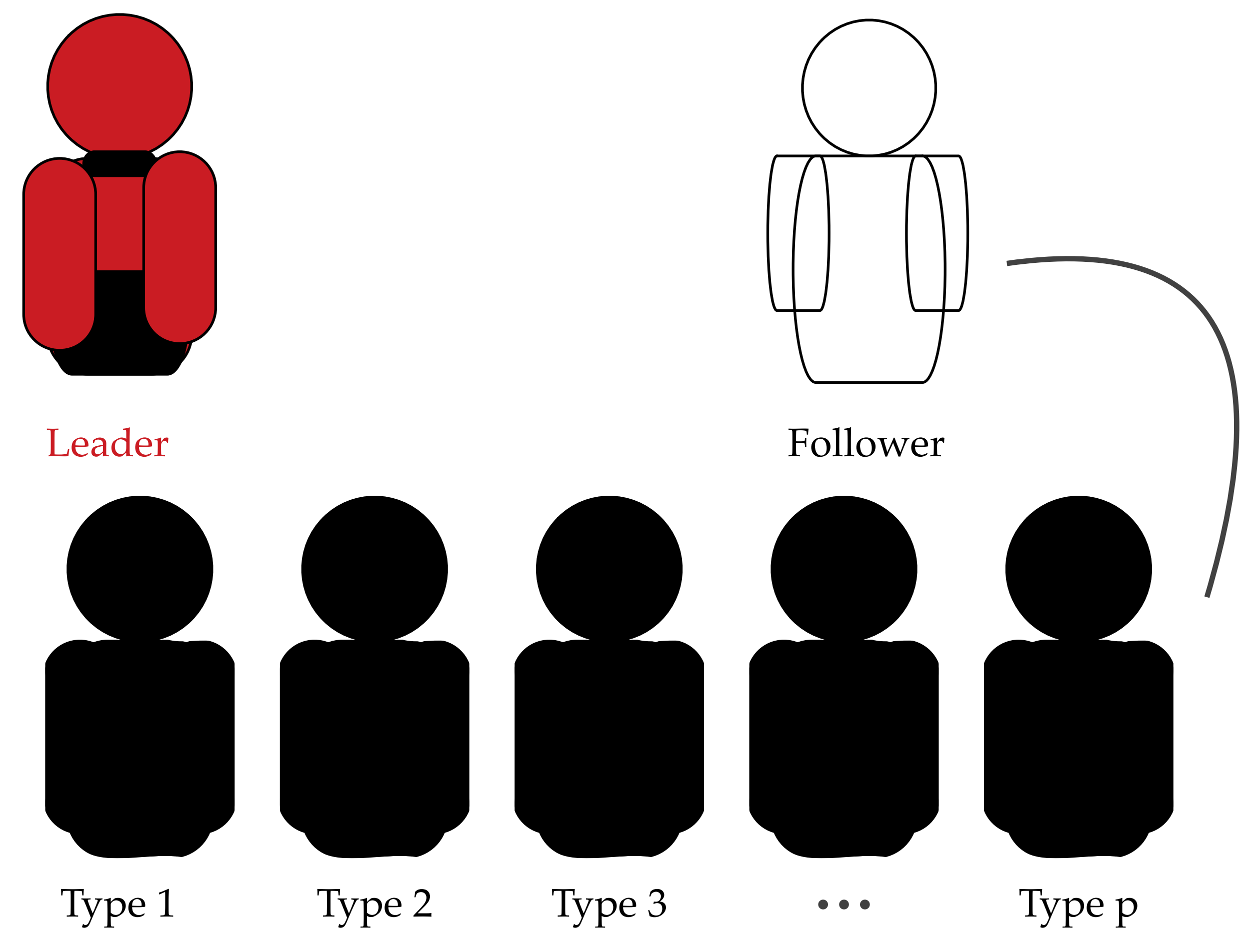
Stackelberg Games

Stackelberg Game



p-Followers Stackelberg Game

(Conitzer and Sandblom, 2006)



Objective of the Game

- Reward-maximizing strategy for the Leader.
- Follower will best respond to observable Leader's strategy.

Bilevel formulation for Stackelberg game

$$
\begin{aligned}
\text{(BIL-}p\text{-G)} \quad \text{Max}_{x,q} \quad & \sum_{i\in I}\sum_{j\in J}\sum_{k\in K} \pi^k R_{ij}^k x_i q_j^k \\
\text{s.t.} \quad & \sum_{i\in I} x_i = 1, \\
& x_i \in [0,1] & \forall i \in I, \\
& q^k = \arg\max_{r^k} \left\{ \sum_{i\in I}\sum_{j\in J} C_{ij}^k x_i r_j^k \right\} & \forall k \in K, \\
& r_j^k \in [0,1] & \forall j \in J, \forall k \in K, \\
& \sum_{j\in J} r_j^k = 1 & \forall k \in K.
\end{aligned}
$$

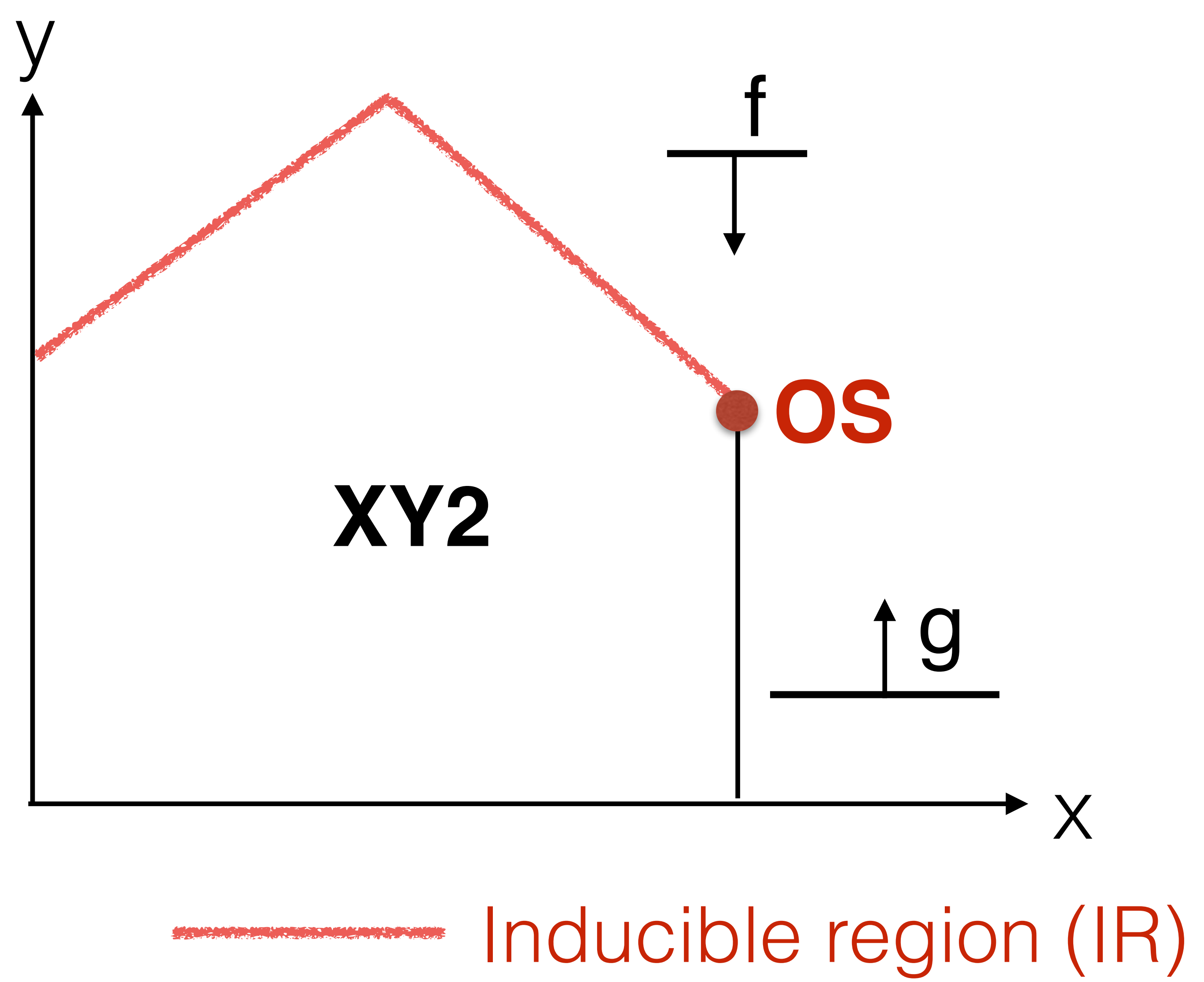
Bilevel optimization: Modeling issues

Linear BP: maybe the simplest ones

$$\begin{aligned}
\min_{x,y} \quad & cx + dy \\
\text{s.t.} \quad & Ax + By \geq a \\
& \min_{y} \quad fy \\
& \text{s.t.} \quad Cx + Dy \geq b
\end{aligned}$$

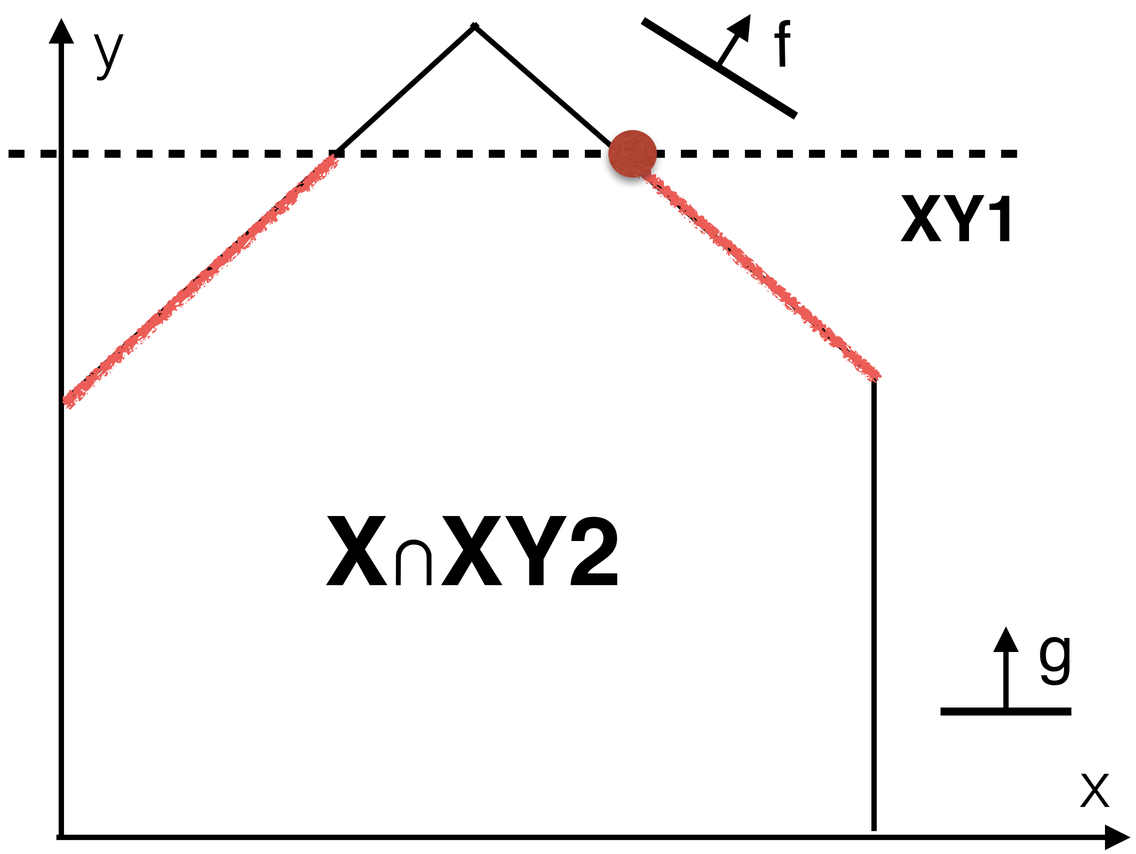
Example of linear BP

$$\max_{x,y} \quad f_1 x + f_2 y$$
$$\text{s.t.} \quad \max_y g_1 x + g_2 y$$
$$\text{s.t.} (x, y) \in XY2$$



Coupling constraints

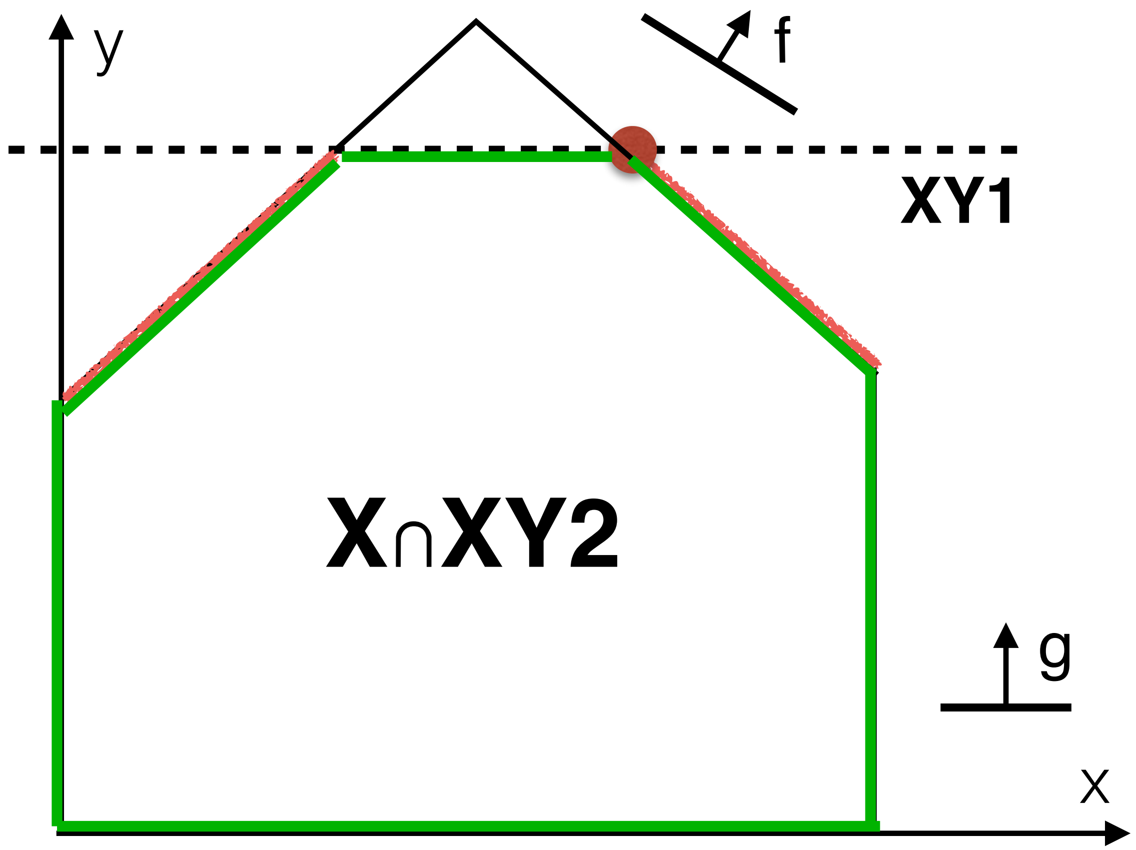
The follower sees only the second level constraints



$$\max_{x,y} \quad f_1 x + f_2 y$$
$$\text{s.t.} \quad x \in X$$
$$(x,y) \in XY1$$
$$\max_{y} g_1 x + g_2 y$$
$$\text{s.t.}(x,y) \in XY2$$

High point relaxation

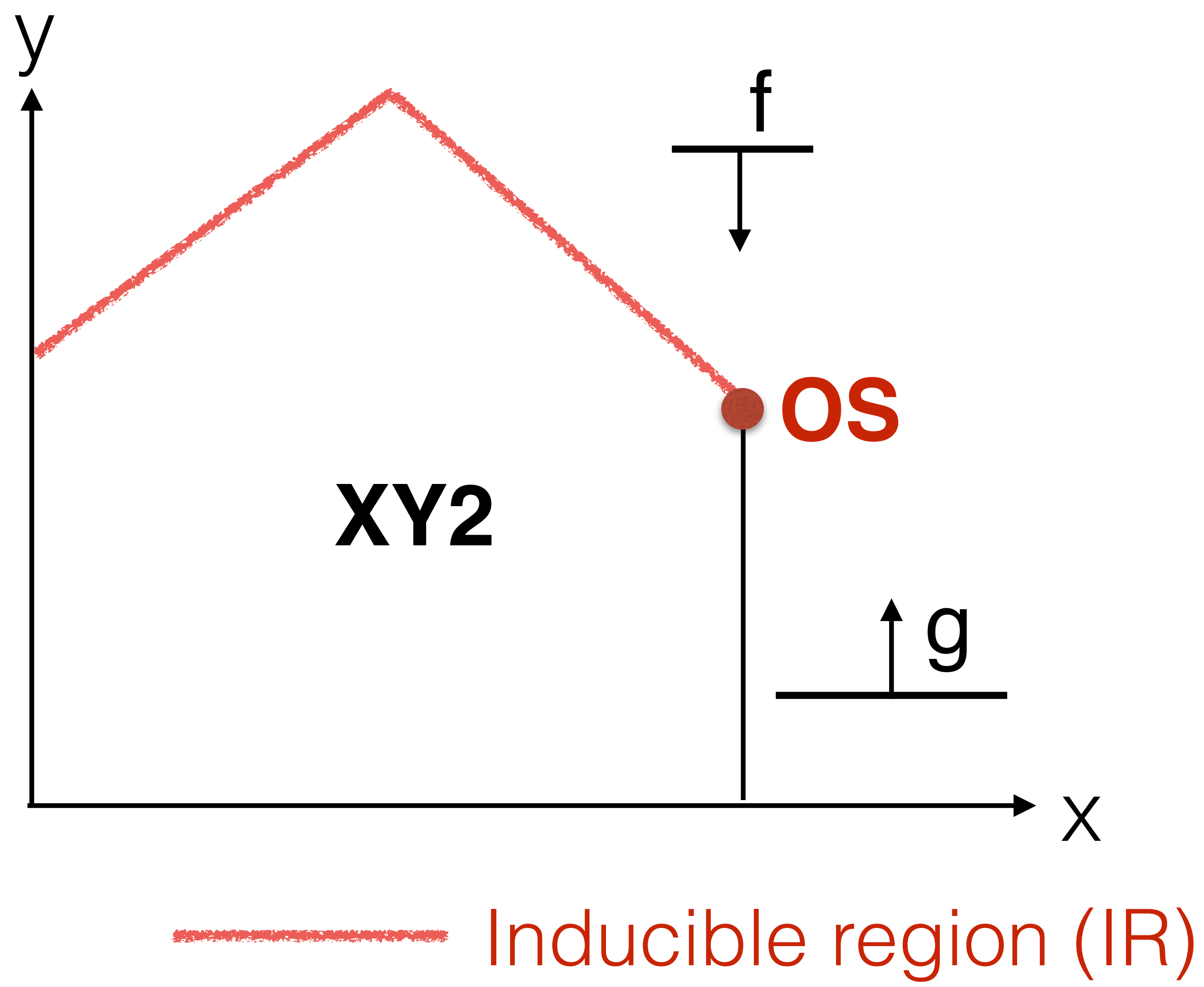
The follower sees only the second level constraints



$$\begin{aligned} \max_{x,y} \quad & f_1 x + f_2 y \\ \text{s.t.} \quad & x \in X \\ & (x,y) \in XY1 \\ & \max_y g_1 x + g_2 y \\ & \text{s.t.} (x,y) \in XY2 \end{aligned}$$

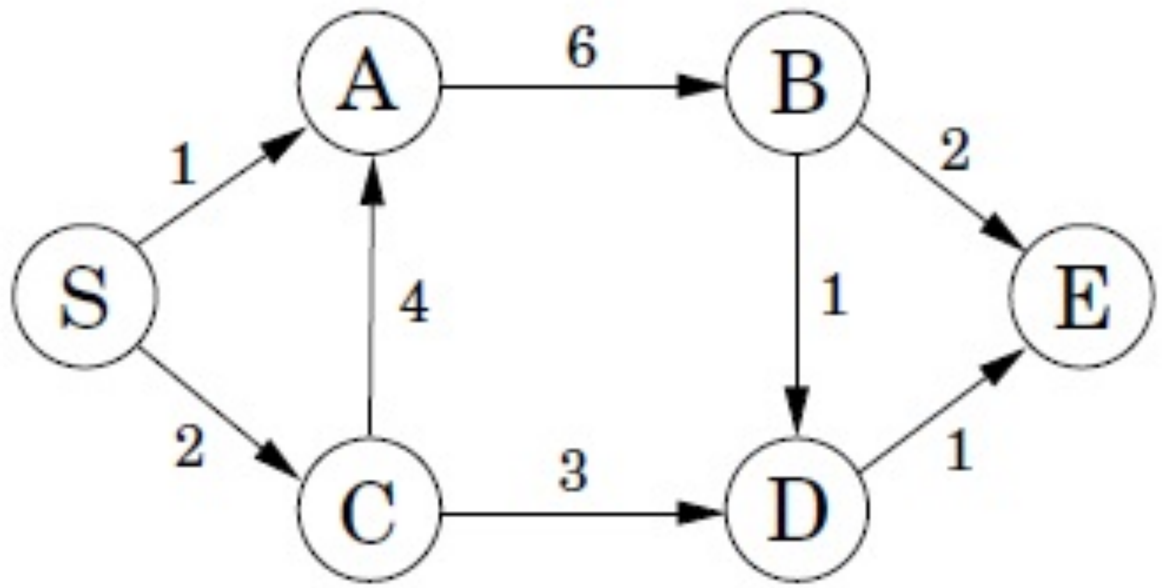
$$X \cap XY1 \cap XY2 = \text{High Point Relaxation (HPR)}$$

High point relaxation



Understanding the model: example

The shortest path interdiction problem



Interdict set S of at most K arcs in order to maximize length of shortest path from s to t in (V, A/S)

- x_{ij} = 1 if (i,j) is interdicted
- y_{ij} = 1 if (i,j) belongs to (shortest) path
- $Ny = \boldsymbol{\delta}$

The shortest path interdiction problem

$$
\begin{aligned}
\max_{x,y} \quad & (c+Mx)y \\
\text{s.t.} \quad & ex \leq K \\
& x \in \{0,1\}^m \\
& \min_y \quad (c+Mx)y \\
& \text{s.t.} \quad Ny = \delta \\
& \qquad\quad y \geq 0
\end{aligned}
$$

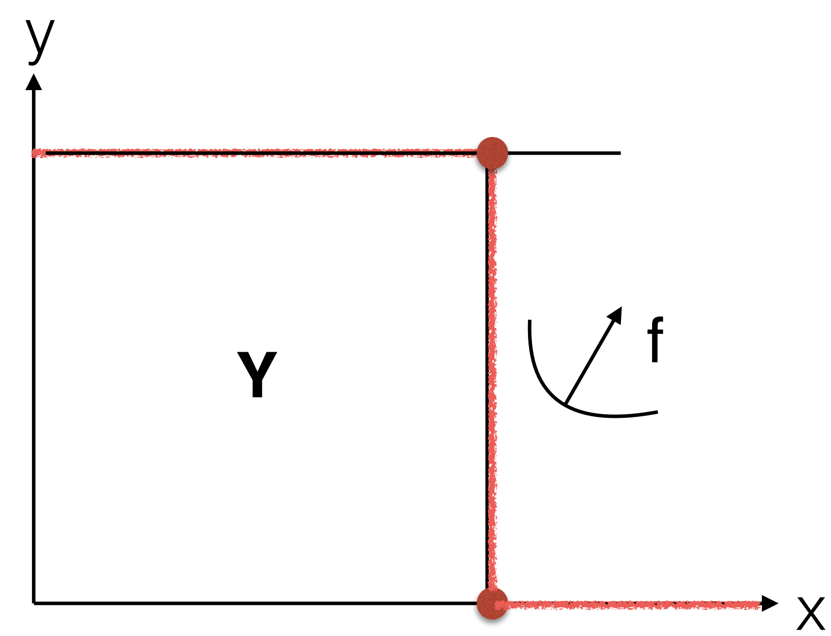
$$
\begin{aligned}
\max_{x,y} \quad & cy \\
\text{s.t.} \quad & ex \leq K \\
& x \in \{0,1\}^m \\
& \min_y \quad cy \\
& \text{s.t.} \quad Ny = \delta \\
& \qquad\quad x + y \leq e \\
& \qquad\quad y \geq 0
\end{aligned}
$$

Multiple second level optima

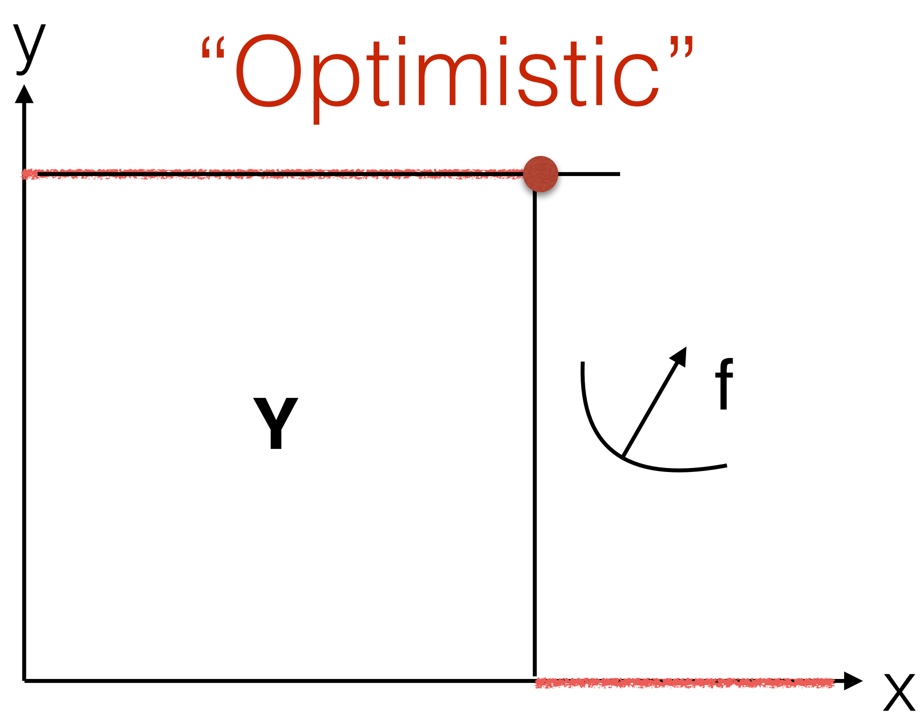
$$\max_{x \geq 0} \quad xy$$

$$\text{s.t.} \quad \max_{y} (1-x)y$$

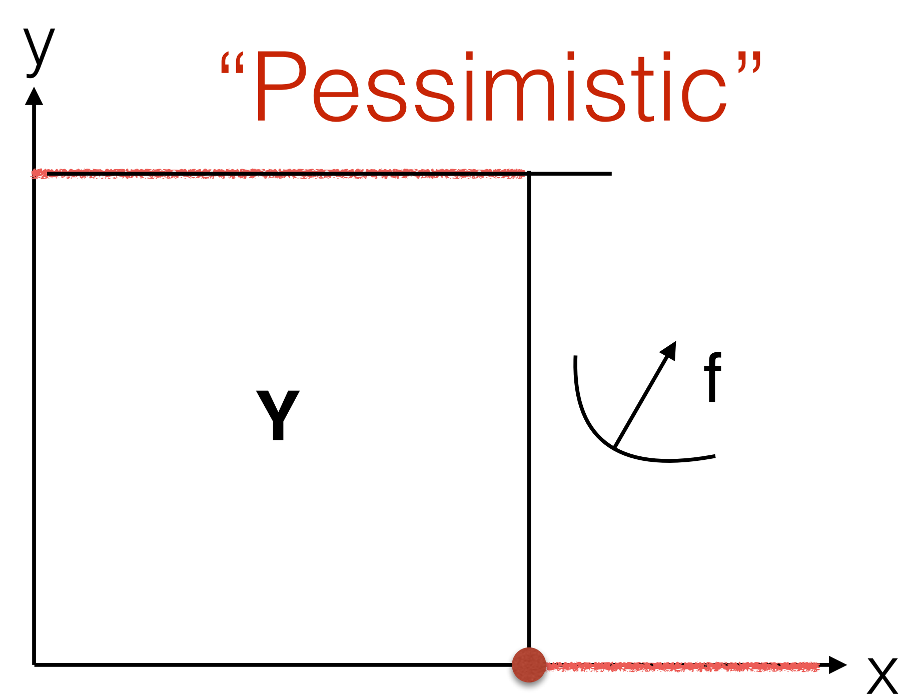
$$\text{s.t.} 0 \leq y \leq 1$$



“Optimistic”



“Pessimistic”



Multiple followers independent

$$\begin{aligned} \max_{x} \quad & f(x, y^1, \ldots, y^n) \\ \text{s.t.} \quad & (x, y^1, \ldots, y^n) \in X \\ & \max_{y^k} g(x, y^k), k = 1, \ldots, n \\ & \text{s.t.} (x, y^k) \in Y^k \end{aligned}$$

Multiple followers dependent

$$\max_{x} \quad f(x, y^1, \ldots, y^n)$$

$$\text{s.t.} \quad (x, y^1, \ldots, y^n) \in X$$

$$\max_{y^k} g(x, y^k, y^{-k}), k = 1, \ldots, n$$

$$\text{s.t.}(x, y^k, y^{-k})) \in Y^k$$

Basic references

- L.N. Vincente and P.H. Calamai (1994), Bilevel and multilevel programming : a bibiliography review, J. Global Optim. 5, 291-306.
- S. Dempe. Foundations of bilevel programming. In Nonconvex optimization and its applications, volume 61. Kluwer Academic Publishers, 2002.
- B. Colson, P. Marcotte, and G. Savard. Bilevel programming: A survey. 4 OR, 3:87-107, 2005.
- M. Labbé and A. Violin. Bilevel programming and price setting problems. 4OR, 11:1-30, 2013.

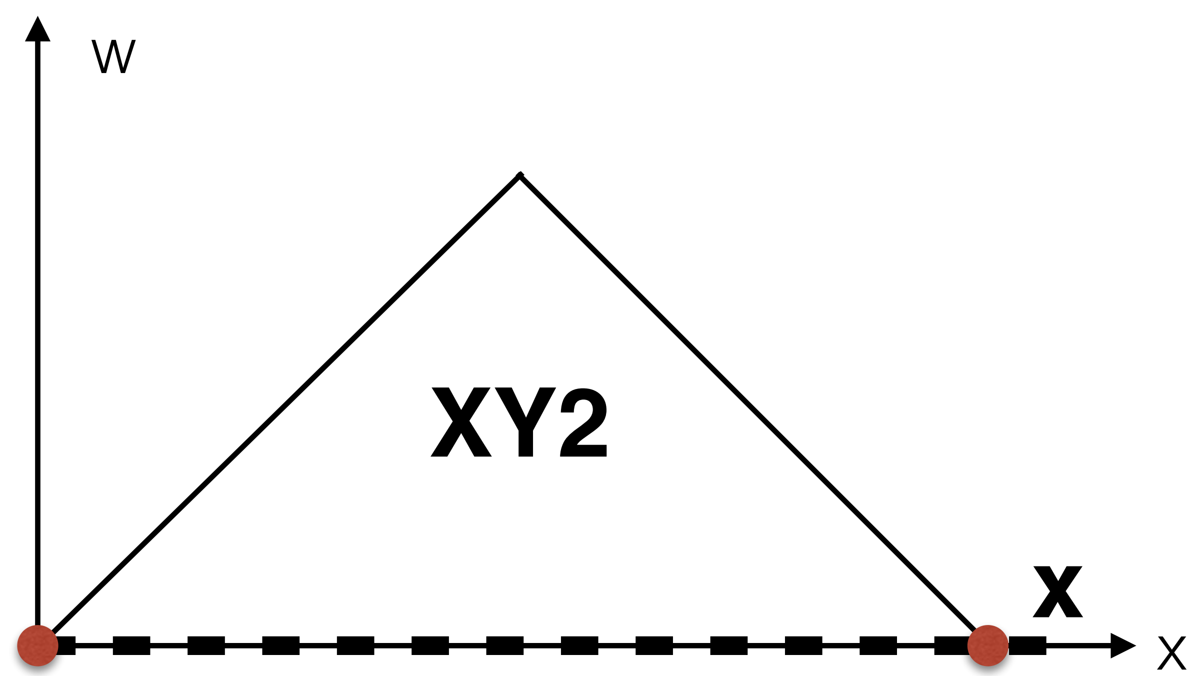
Linear bilevel problems:

Linear BP

$$\begin{aligned} \min_{x,y} \quad & cx + dy \\ \text{s.t.} \quad & Ax + By \geq a \\ & \min_{y} \quad fy \\ & \text{s.t.} \quad Cx + Dy \geq b \end{aligned}$$

0/1 Programming is a special case of BLO (Audet et al. 1997)

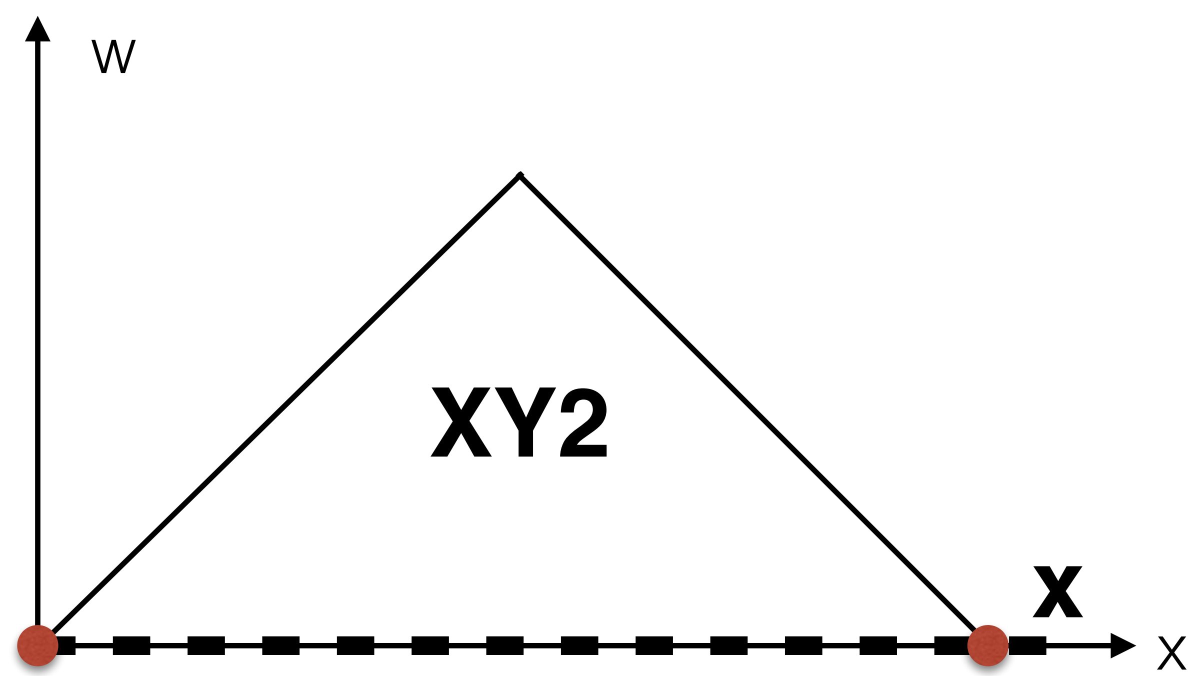
$$x \in \{0,1\} \Leftrightarrow v = 0 \text{ and } v = \underset{w}{\operatorname{argmax}}\{w : w \leq x, w \leq 1 - x, w \geq 0\}$$



0/1 Programming is a special case of BLO
(Audet et al. 1997)

$$x \in \{0,1\} \Leftrightarrow \boxed{v = 0} \text{ and } v = \underset{w}{\text{argmax}}\{w : w \leq x, w \leq 1-x, w \geq 0\}$$

Coupling constraint



Linear BP

- Linear BP is strongly NP-hard (Hansen et al. 1992): Kernel reduces to linear BP.

- MILP is a special case of Linear BP

Linear BP

(Bialas & Karwan(1982), Bard(1983)).

- IR is the union of faces of HPR
- IR is connected if there is no coupling constraint
- If Linear BP is feasible, then there exists an optimal solution which is a vertex of HPR.

→ K-th best algorithm

Linear BP- single level reformulation

$$\begin{aligned} \min_x \quad & cx + dy \\ \text{s.t.} \quad & Ax + By \geq a \\ & \min_y fy, \\ & \text{s.t.} \quad Dy \geq b - Cx \quad (\lambda) \end{aligned}$$

$$\begin{aligned} \min_x \quad & cx + dy \\ \text{s.t.} \quad & Ax + By \geq a \\ & Dy \geq b - Cx \\ & \lambda D = f \\ & \boxed{\lambda(Dy - b + Cx) = 0} \\ & \lambda \geq 0 \end{aligned}$$

- Branch & Bound (Hansen et al.1992)
- Branch & Cut (Audet et al. 2007)

Linear BP

$$
\begin{aligned}
\min_{x} \quad & cx + dy \\
\text{s.t.} \quad & Ax + By \geq a \\
& Dy \geq b - Cx \\
& \lambda D = f \\
& \lambda(Dy - b + Cx) = 0 \\
& \lambda \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\min_{x} \quad & cx + dy \\
\text{s.t.} \quad & Ax + By \geq a \\
& Cx + Dy \geq b \\
& \lambda D = f \\
& \lambda \geq 0 \\
& \lambda \leq M_d z \\
& Cx + Dy \leq b + M_p(1 - z) \\
& z \in \{0, 1\}^m
\end{aligned}
$$

Fortuny-Amat, McCarl (1981)

Linear BP

$$
\begin{aligned}
\min_{x} \quad & cx + dy \\
\text{s.t.} \quad & Ax + By \geq a \\
& Cx + Dy \geq b \\
& \lambda D = f \\
& \lambda \geq 0 \\
& \lambda \leq M_d z \\
& Cx + Dy \leq b + \boxed{M_p}(1 - z) \\
& z \in \{0, 1\}^m
\end{aligned}
$$

Often available

Linear BP: valid big M

Trial-and-error tuning procedure:

- Choose some arbitrary values for M_p and M_d
- Solve the MIP
- If some M_p or M_d appear in “active” constraints, increase them and iterate.

Linear BP: valid big M

Pineda and Morales (2018): This trial-and-error method is wrong

$$
\begin{aligned}
\max_{x \in \mathbb{R}} \quad & z = x + y \\
\text{s.t.} \quad & 0 \leq x \leq 2 \\
& \min_{y \in \mathbb{R}} \quad y \\
& \quad \text{s.t.} \quad y \geq 0 \quad (\lambda_1) \\
& \qquad\quad x - 0.01y \leq 1 \quad (\lambda_2)
\end{aligned}
$$

$$
\begin{aligned}
\max_{x \in \mathbb{R}, y \in \mathbb{R}} \quad & z = x + y \\
\text{s.t.} \quad & 0 \leq x \leq 2 \\
& y \geq 0 \\
& x - 0.01y \leq 1 \\
& 1 - \lambda_1 - 0.01\lambda_2 = 0 \\
& \lambda_1, \lambda_2 \geq 0 \\
& \lambda_1 \leq u_1 M_1^D \\
& y \leq (1 - u_1) M_1^P \\
& \lambda_2 \leq u_2 M_2^D \\
& -x + 0.01y + 1 \leq (1 - u_2) M_2^P \\
& u_1, u_2 \in \{0, 1\}
\end{aligned}
$$

Linear BP: valid big M

$$
\begin{aligned}
\max_{x\in\mathbb{R}, y\in\mathbb{R}} \quad & z = x + y \\
\text{s.t.} \quad & 0 \le x \le 2 \\
& y \ge 0 \\
& x - 0.01y \le 1 \\
& 1 - \lambda_1 - 0.01\lambda_2 = 0 \\
& \lambda_1, \lambda_2 \ge 0 \\
& \lambda_1 \le u_1 M_1^D \\
& y \le (1 - u_1) M_1^P \\
& \lambda_2 \le u_2 M_2^D \\
& -x + 0.01y + 1 \le (1 - u_2) M_2^P \\
& u_1, u_2 \in \{0, 1\}
\end{aligned}
$$

D2 is bounded: $\lambda_1 \le 1, \lambda_2 \le 100$

The trial-and-error procedure stops for

$M_1^P = M_2^P = 200$, $M_1^D = M_1^D = 50$

with a solution that is not optimal.

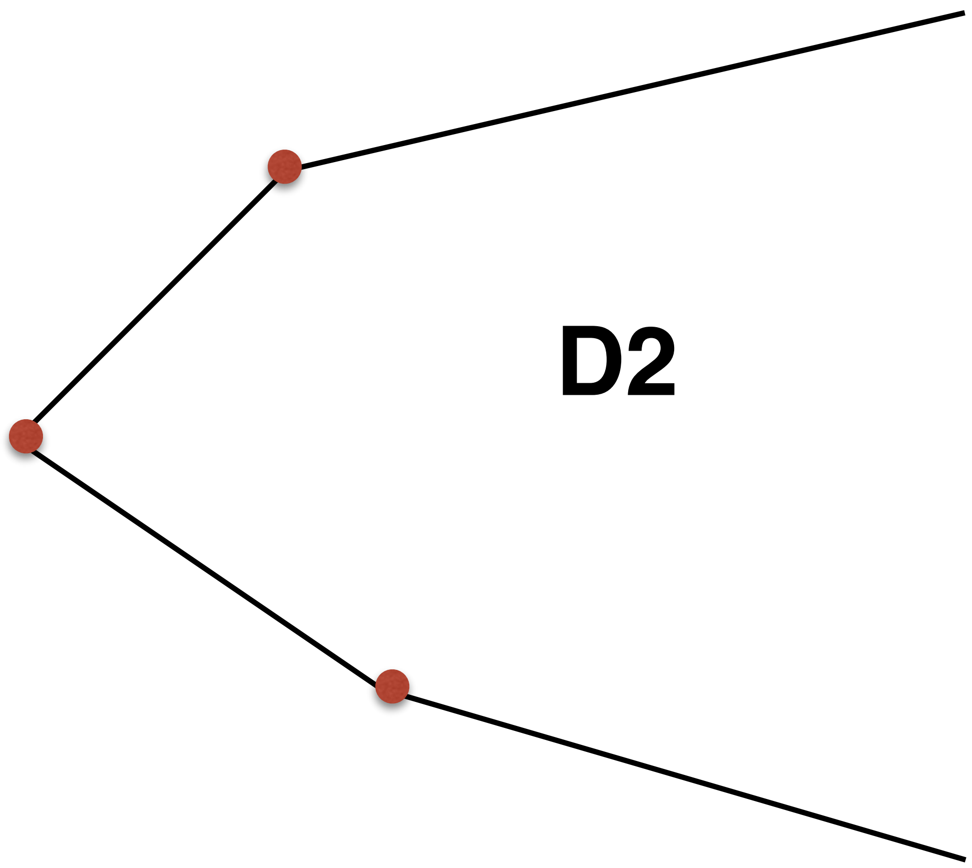
Linear BP: valid big M_d

- Upper bound on any $\boldsymbol{\lambda}$ in D2= $\{\lambda \geq 0 : \lambda D = f\}$

- D2 is often unbounded

- In any optimal solution (x^*,y^*, $\boldsymbol{\lambda}^*$) to LBP,
- $\boldsymbol{\lambda}^*$ is the optimal solution of an LP on D2
- $\boldsymbol{\lambda}^*$ is a vertex of D2

Linear BP: valid big M_d



Linear BP: valid big M_d

(Kleinert, Labbé, Schmidt, Plein 2020)

Problem valid M_d: given A, b, M

Question: is $x_i \leq M$ for every vertex x of $Ax \leq b$?

Strongly co-NP complete

Its complement:

Question: does it exists a vertex x of $Ax \leq b$ such that $x_i \leq M$?

Strongly NP complete

Linear BP: reformulation using strong duality

$$
\begin{aligned}
\min_x \quad & cx + dy \\
\text{s.t.} \quad & Ax + By \geq a \\
& \min_y fy, \\
& \text{s.t.} \quad Dy \geq b - Cx \quad (\lambda)
\end{aligned}
$$

$$
\begin{aligned}
\min_x \quad & cx + dy \\
\text{s.t.} \quad & Ax + By \geq a \\
& Dy \geq b - Cx \\
& \lambda D = f \\
& \lambda \geq 0 \\
& fy \leq \lambda(b - Cx)
\end{aligned}
$$

Linear BP: a valid Primal-Dual inequality

(Kleinert, Labbé, Schmidt, Plein)

$$\boxed{fy + \lambda Cx - \lambda b \leq 0}$$

$$C_i^- = \min_x C_{i.}x \text{ s.t. } (x, y, \lambda) \text{ is 'feasible'}$$

$\Rightarrow$ Valid linear inequality: $fy + C^- x - \lambda b \leq 0$

Linear BP: a valid Primal-Dual inequality

(Kleinert, Labbé, Schmidt, Plein)

$$fy + \lambda Cx - \lambda b \leq 0$$

McCormick (1976) envelopes: $z_i = \lambda_i (C_{i.} x)$

$$fy + \sum_i z_i - \lambda b \leq 0$$

$$z_i \geq \lambda_i^+ C_{i.} x + \lambda_i C_i^+ - \lambda_i^+ C_i^+$$
$$z_i \geq \lambda_i^- C_{i.} x + \lambda_i C_i^- - \lambda_i^- C_i^-$$

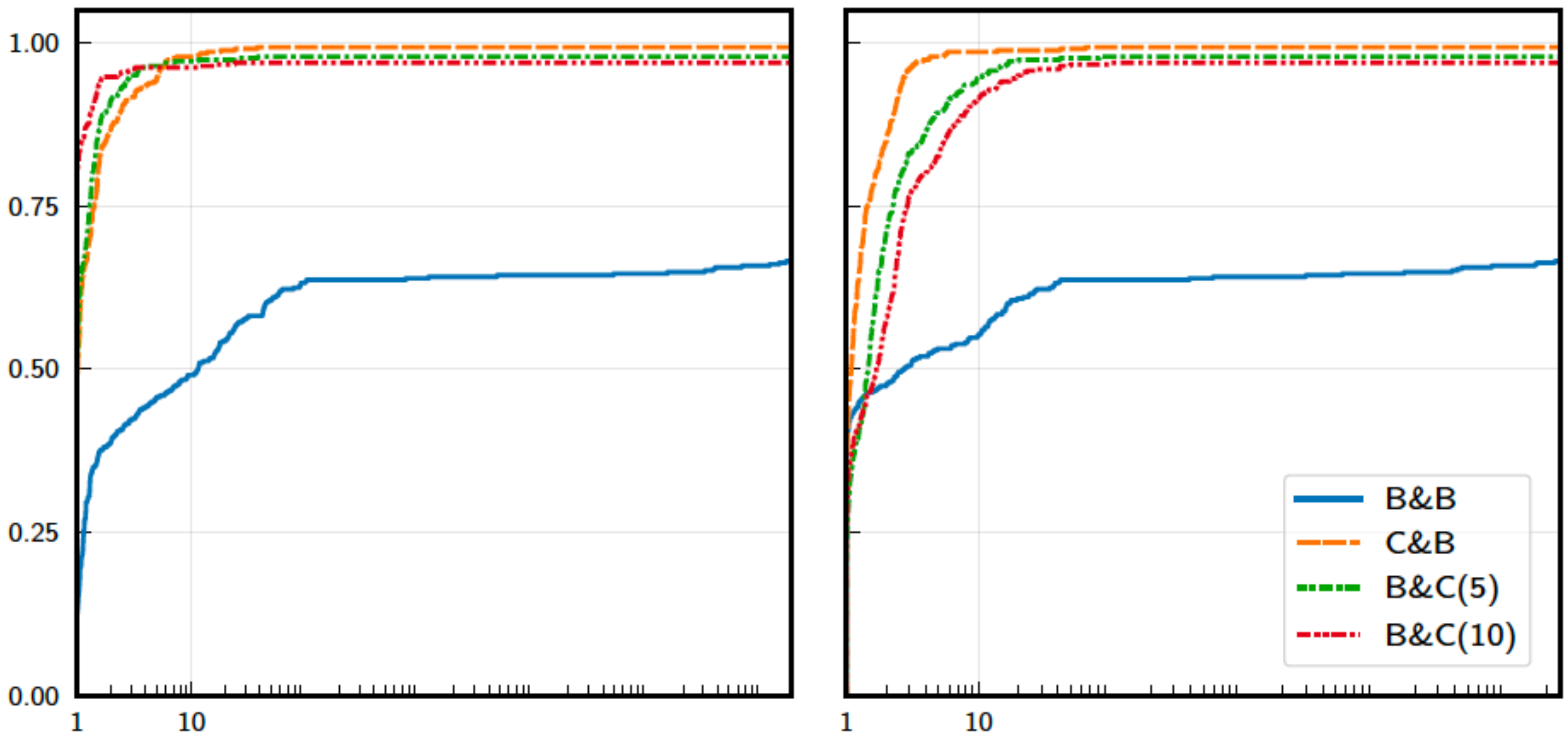


FIGURE 4. Log-scaled performance profiles for branch-and-bound nodes (left) and running times (right) over all remaining instances.

Bilevel problems with bilinear objectives and linear constraints

Adequate framework for Price Setting Problem

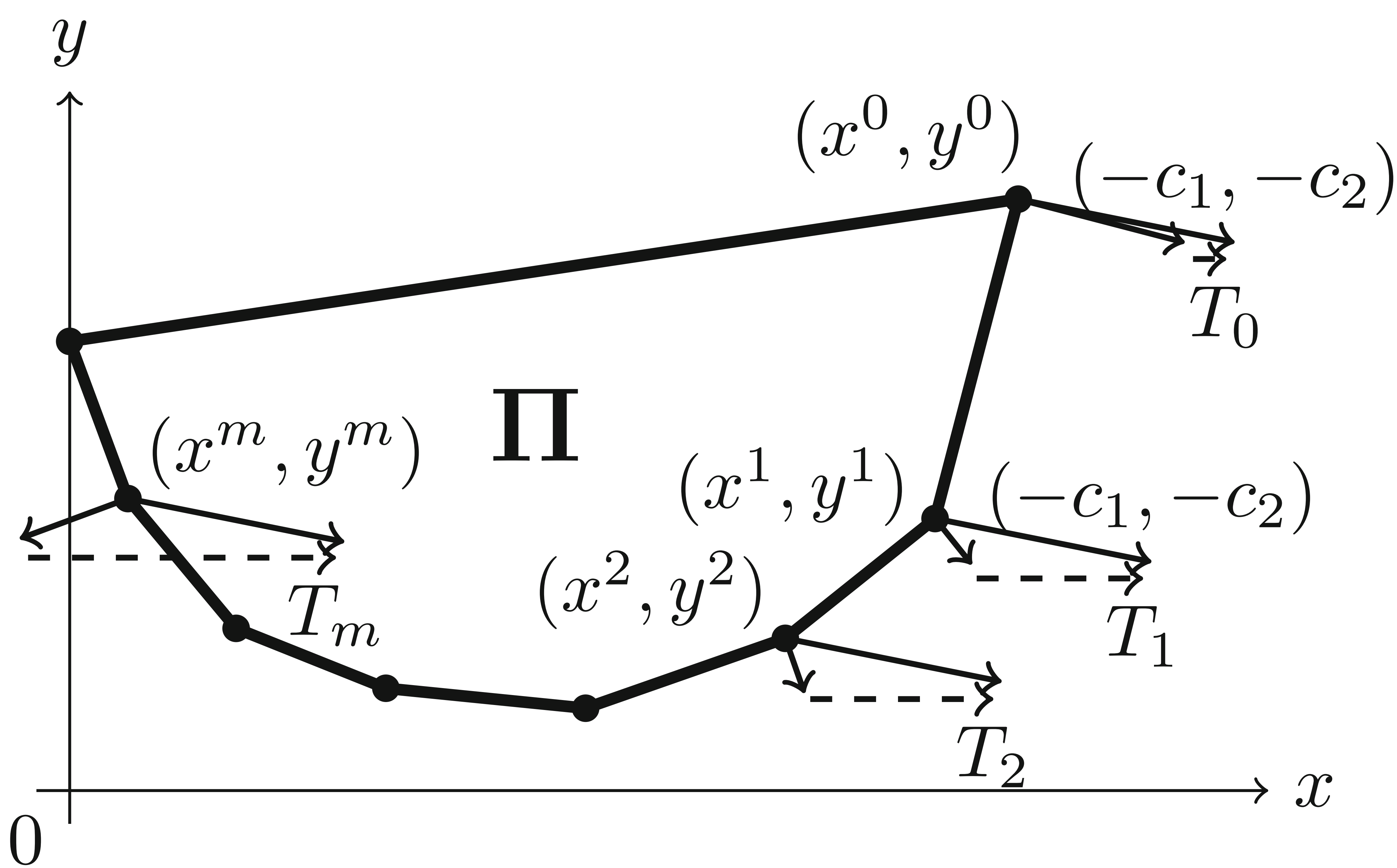
$$\begin{aligned} \max_{T \in \Theta, x, y} \quad & F(T, x, y) \\ \text{s.t.} \quad & \min_{x, y} f(T, x, y) \\ & \text{s.t.} (x, y) \in \Pi \end{aligned}$$

Price Setting Problem with linear constraints

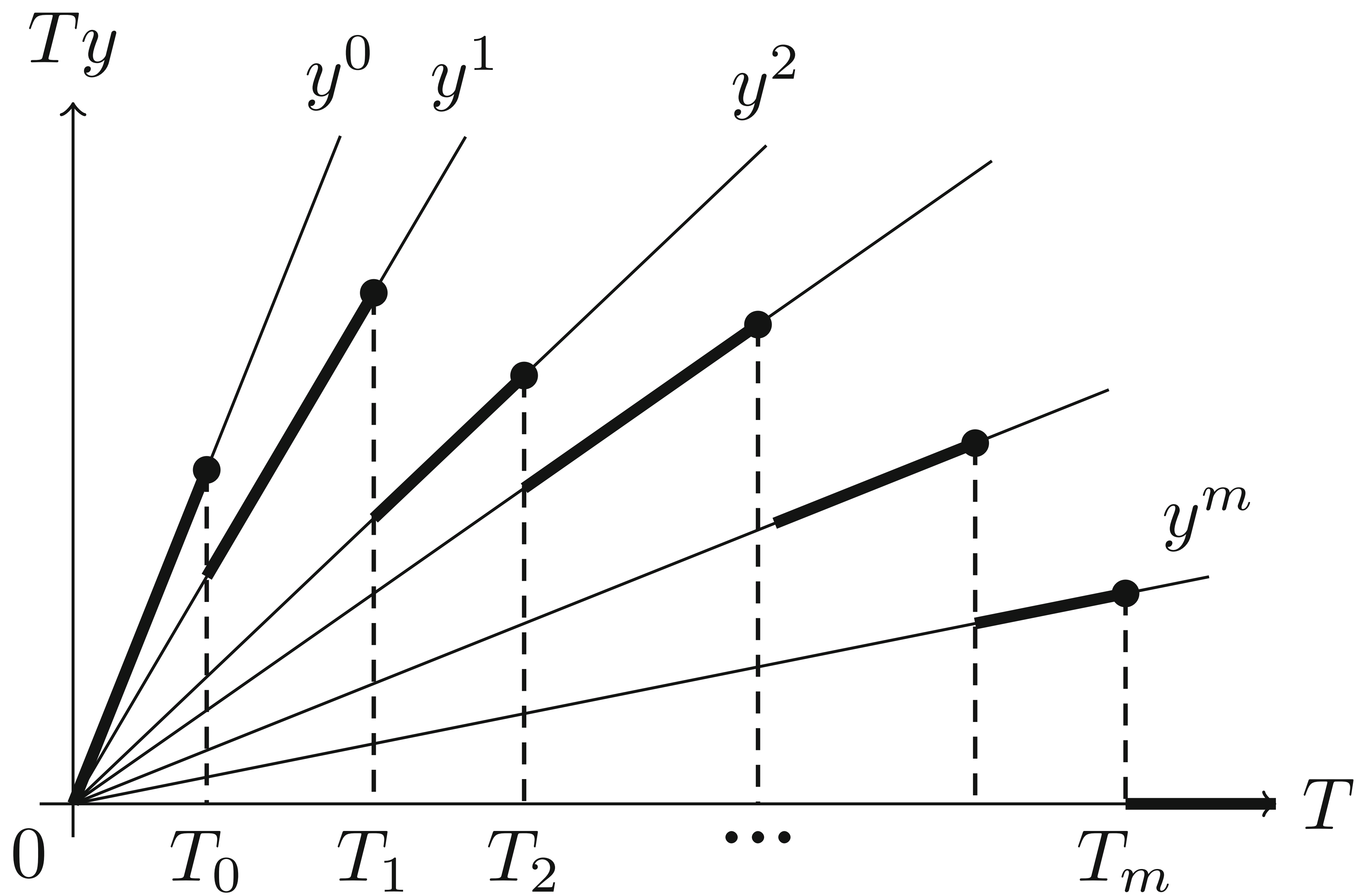
$$\begin{aligned} \max_{T,x,y} \quad & Tx \\ \text{s.t.} \quad & TC \geq f \\ \min_{x,y} \quad & (c+T)x + dy \\ \text{s.t.} \quad & Ax + By \geq b \end{aligned}$$

- $\Pi = \{x, y : Ax + By \geq b\}$ is bounded
- $\{(x, y) \in \Pi : x = 0\}$ is nonempty

Example: 2 variables in second level



The first level revenue



Reformulation

$$
\begin{aligned}
\max_{T} \quad & Ty_1 \\
\text{s.t.} \quad & \min_{y_1,y_2} (c_1 + T)y_1 + c_2 y_2 \\
& \text{s.t.} A_1 y_1 + A_2 y_2 = b \\
& y_1, y_2 \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\max_{T} \quad & Ty_1 & \\
\text{s.t.} \quad & A_1 y_1 + A_2 y_2 & = b \\
& y_1, y_2 & \geq 0 \\
& \lambda A_1 & \leq c_1 + T \\
& \lambda A_2 & \leq c_2 \\
& (c_1 + T)y_1 + c_2 y_2 & = \lambda b
\end{aligned}
$$

More reformulation

$$
\begin{aligned}
\max_{T} \quad & T y_1 & \\
\text{s.t.} \quad & A_1 y_1 + A_2 y_2 & = b \\
& y_1, y_2 & \geq 0 \\
& \lambda A_1 & \leq c_1 + T \\
& \lambda A_2 & \leq c_2 \\
& (c_1 + T) y_1 + c_2 y_2 & = \lambda b
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & \lambda b - (c_1 y_1 + c_2 y_2) & \\
\text{s.t.} \quad & A_1 y_1 + A_2 y_2 & = b \\
& y_1, y_2 & \geq 0 \\
& \lambda A_1 & \leq c_1 + T \\
& (c_1 + T - \lambda A_1) y_1 & = 0 \\
& \lambda A_2 & \leq c_2 \\
& (c_2 - \lambda A_2) y_2 & = 0
\end{aligned}
$$

More reformulation

$$
\begin{aligned}
\max_{T} \quad & T y_1 \\
\text{s.t.} \quad & A_1 y_1 + A_2 y_2 && = b \\
& y_1, y_2 && \geq 0 \\
& \lambda A_1 && \leq c_1 + T \\
& \lambda A_2 && \leq c_2 \\
& (c_1 + T) y_1 + c_2 y_2 && = \lambda b
\end{aligned}
$$

$$
\begin{aligned}
\max \quad & \lambda b - (c_1 y_1 + c_2 y_2) \\
\text{s.t.} \quad & A_1 y_1 + A_2 y_2 && = b \\
& y_1, y_2 && \geq 0 \\
& \lambda A_1 && \leq c_1 + T \\
& (c_1 + T - \lambda A_1) y_1 && = 0 \\
& \lambda A_2 && \leq c_2 \\
& (c_2 - \lambda A_2) y_2 && = 0
\end{aligned}
$$

Reformulation-Interpretation

$$\max_{y_1, y_2, \lambda} \quad \lambda b \quad - \quad (c_1 y_1 + c_2 y_2)$$

$$s.t. \quad \lambda A_2 \le c_2 \qquad A_1 y_1 + A_2 y_2 = b$$

$$(c_2 - \lambda A_2) y_2 = 0$$

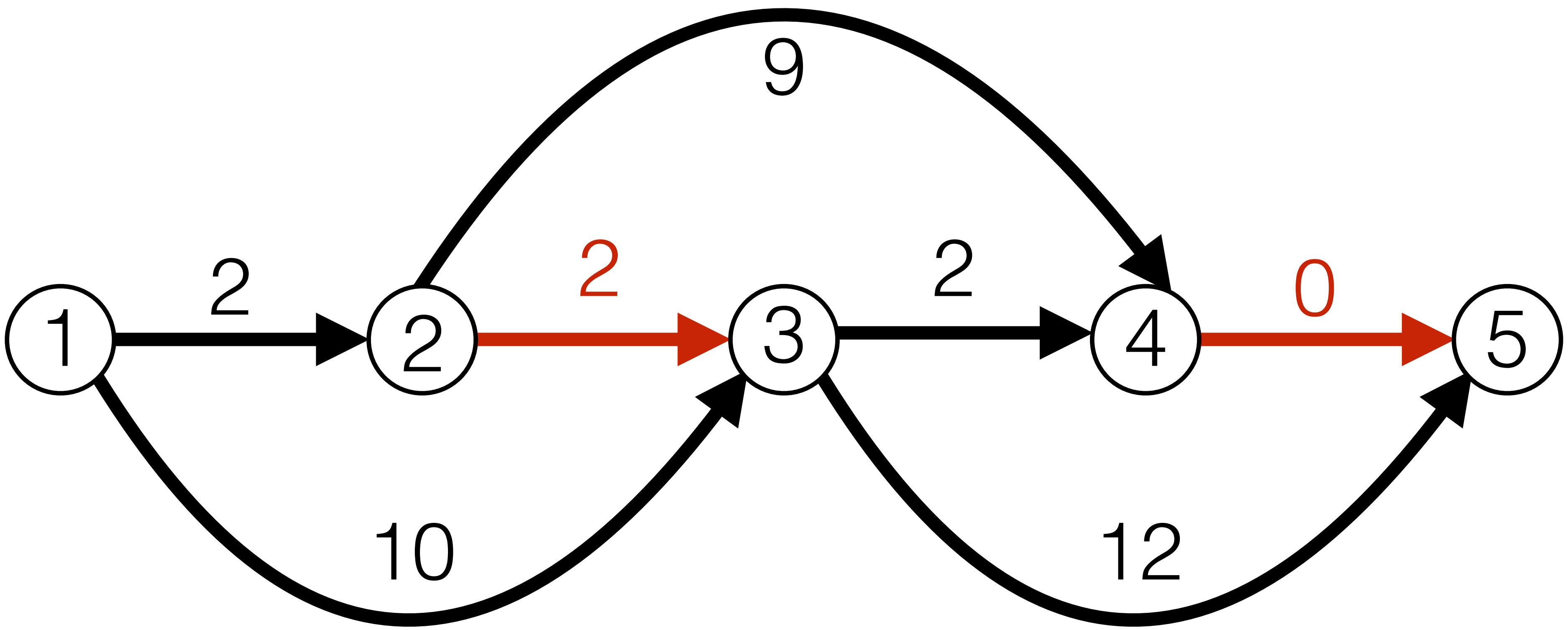
Dual of lower level with infinite taxes

Lower level with zero taxes

Network pricing problem
(Labbé, Marcotte, Savard, 1998)

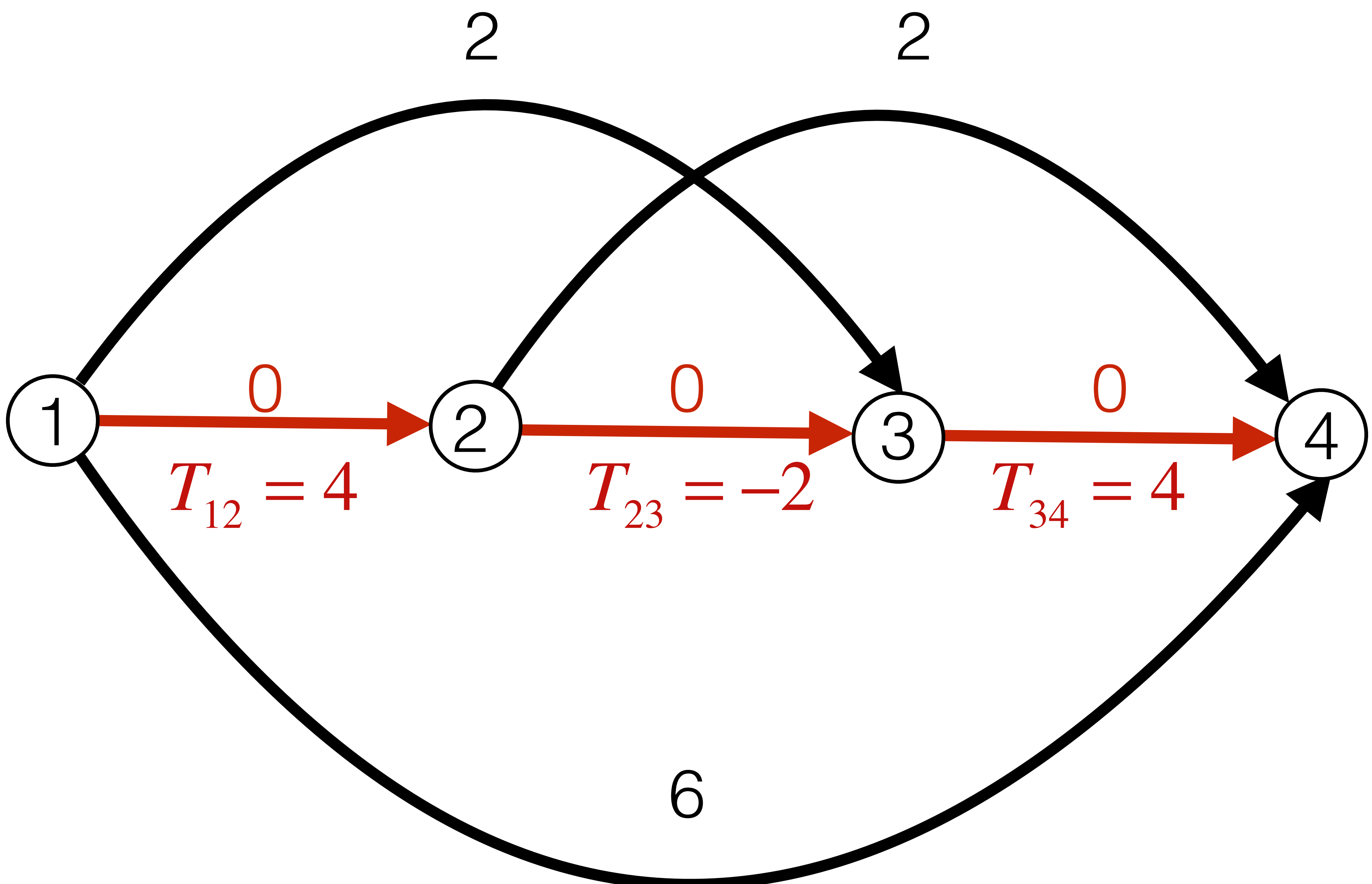
- network with toll arcs (A_1) and non toll arcs (A_2)
- Costs c_a on arcs
- Commodities (o^k, d^k, n^k)
- Routing on cheapest (cost + toll) path
- Maximize total revenue

Example

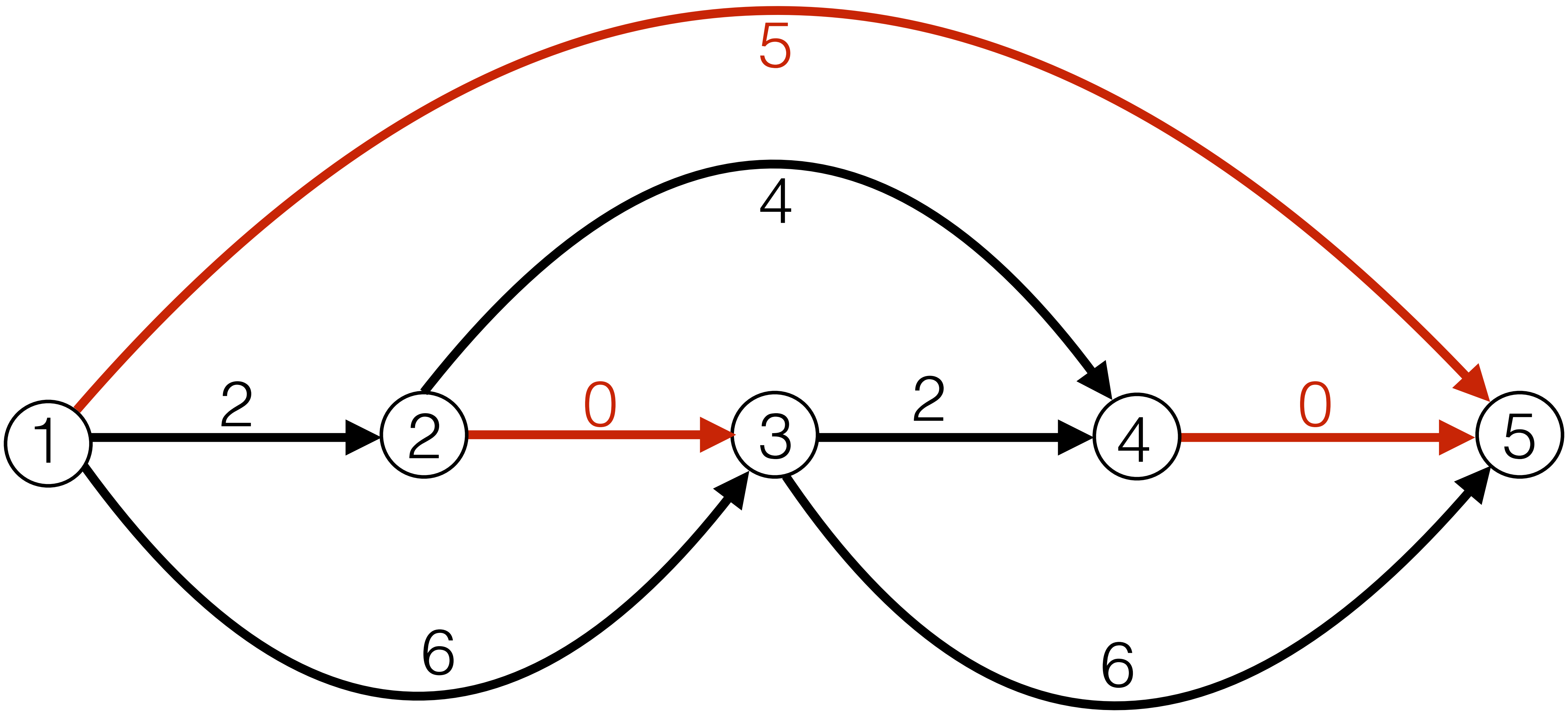


- $\text{UB on } (T_1 + T_2) = \text{SPL}(T = \infty) - \text{SPL}(T = 0) = 22 - 6 = 16$
- $T_{2,3} = 5, T_{4,5} = 10$

Example with negative toll arc



Example with flow not assigned to SP(T=0)



$T_{23} = T_{45} = T_{15} = 7$, profit = 7

Network pricing problem

(Labbé et al., 1998, Roch at al., 2005)

- Strongly NP-hard even for only one commodity.
- Polynomial for
  - one commodity if lower level path is known
  - one commodity if toll arcs with positive flows are known
  - one single toll arc.
- Polynomial algorithm with worst-case guarantee of $(\log |A1|)/2 + 1$

One toll arc

- For each k, compute $UB(k)$ on profit is k uses toll arc: $UB(k) = SPC_k(T = \infty) - SPC_k(T = 0)$
- $UB(1) \geq UB(2) \geq \ldots UB(K)$
- $T_a = UB(i^*), UB(i^*) = \text{argmax}\, UB(i) \sum_{k \leq i} n^k$

Network pricing problem

$$\max_{T \geq 0} \quad \sum_{a \in A_1} T_a \sum_{k \in K} n^k x_a^k$$

$$\min_{x,y} \quad \sum_{k \in K} \Big(\sum_{a \in A_1} (c_a + T_a) x_a^k + \sum_{a \in A_2} c_a y_a \Big)$$

$$\text{s.t.} \quad \sum_{a \in i^+} (x_a^k + y_a^k) - \sum_{a \in i^-} (x_a^k + y_a^k) = b_i^k \quad \forall k, i$$

$$x_a^k, y_a^k \geq 0, \quad \forall k, a$$

NPP: single level reformulation

$$\max_{T,x,y,\lambda} \quad \sum_{k \in K} n^k \sum_{a \in A_k} T_a x_a^k$$

$$\text{s.t.} \quad \sum_{a \in i^+} (x_a^k + y_a^k) - \sum_{a \in i^-} (x_a^k + y_a^k) = b_i^k \quad \forall k, i$$

$$\lambda_i^k - \lambda_j^k \leq c_a + T_a \quad \forall k, a \in A_1, i, j$$

$$\lambda_i^k - \lambda_j^k \leq c_a \quad \forall k, a \in A_2, i, j$$

$$\sum_{a \in A_1} (c_a + T_a) x_a^k + \sum_{a \in A_2} c_a y_a = \lambda_{o^k}^k - \lambda_{d^k}^k \quad \forall k$$

$$x_a^k, y_a^k \geq 0 \quad \forall k, a$$

$$T_a \geq 0 \quad \forall a \in A_1$$

NPP: single level reformulation

$$\max_{T,x,y,\lambda} \quad \sum_{k\in K} n^k \sum_{a\in A_k} T_a x_a^k$$

$$\text{s.t.} \quad \sum_{a\in i^+} (x_a^k + y_a^k) - \sum_{a\in i^-} (x_a^k + y_a^k) = b_i^k \quad \forall k, i$$

$$\lambda_i^k - \lambda_j^k \leq c_a + T_a \quad \forall k, a \in A_1, i, j$$

$$\lambda_i^k - \lambda_j^k \leq c_a \quad \forall k, a \in A_2, i, j$$

$$\sum_{a\in A_1} (c_a + T_a) x_a^k + \sum_{a\in A_2} c_a y_a = \lambda_{o^k}^k - \lambda_{d^k}^k \quad \forall k$$

$$x_a^k, y_a^k \geq 0 \quad \forall k, a$$

$$T_a \geq 0 \quad \forall a \in A_1$$

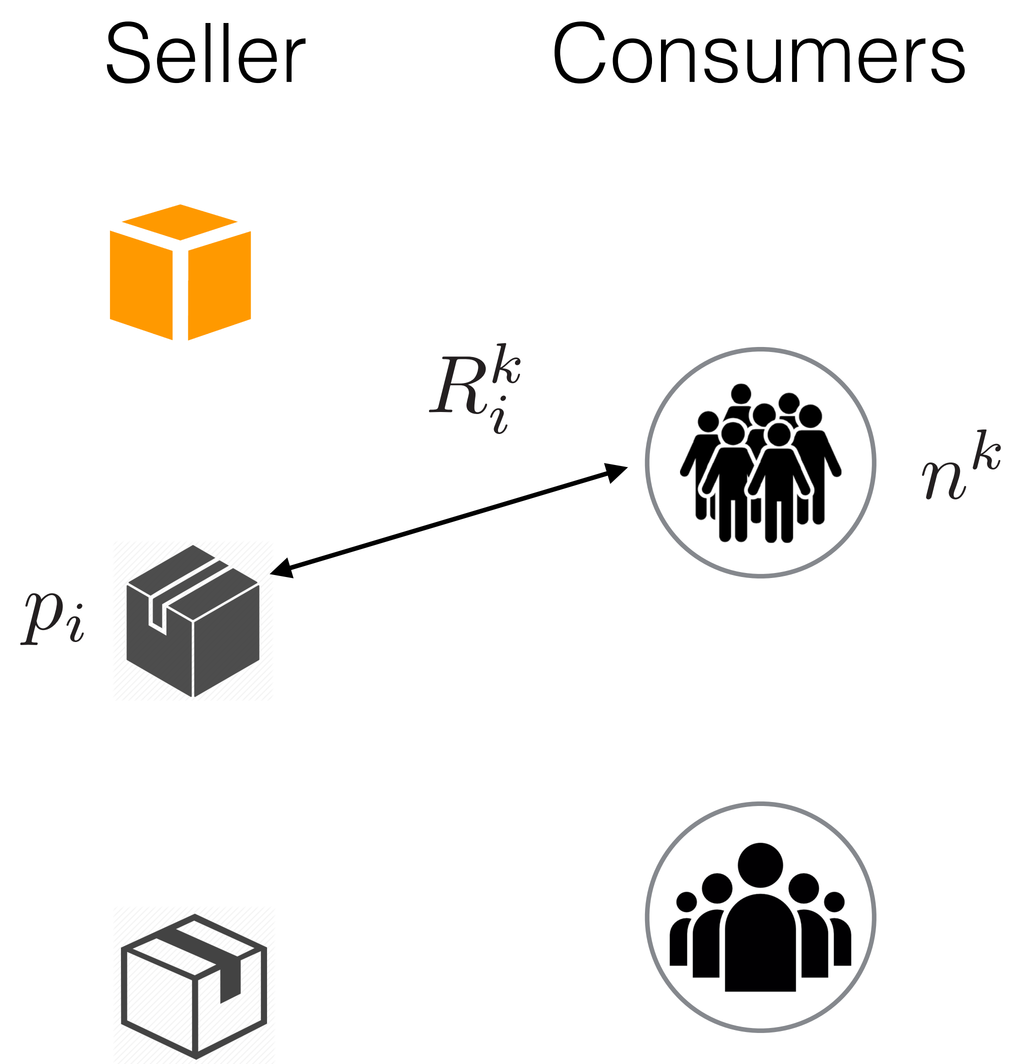
NPP: obtaining a MIP

$$
\begin{aligned}
T_a x_a^k &= p_a^k \\
p_a^k &\leq M_a^k x_a^k \\
T_a - p_a^k &\leq N_a(1 - x_a^k) \\
p_a^k &\leq T_a \\
& x_a^k \in \{0,1\}
\end{aligned}
$$

Solution approach by Branch & Cut

- Formulate NPP as MIP
- Tight bound (M_a^k,N_a) on tax, if arc used and if arc not used, very effective
- Add valid inequalities to strengthen LP relaxation

Product pricing



R_i^k is the reservation price of consumer k for product i

PPP - bilevel formulation

$$
\begin{aligned}
\max_{p \geq 0} \quad & \sum_{k \in K} n^k \sum_{i \in I} p_i x_i^k \\
\text{s.t.} \quad & \max_{x^k} \sum_{i \in I} (R_i^k - p_i) x_i^k, \quad k \in K \\
& \text{s.t.} \quad \sum_{i \in I} x_i^k \leq 1 \\
& \qquad\quad x_i^k \geq 0
\end{aligned}
$$

Product pricing

- PPP is Strongly NP-hard even if reservation price is independent of product (Briest 2006)

- PPP is polynomial for one product or one customer.

PPP - single level formulation

$$
\begin{aligned}
\max_{p \geq 0} \quad & \sum_{k \in K} n^k \sum_{i \in I} p_i x_i^k \\
\text{s.t.} \quad & \sum_{i \in I} (R_i^k - p_i) x_i^k \geq R_j^k - p_j, \quad j \in I, k \in K \\
& \sum_{i \in I} (R_i^k - p_i) x_i^k \geq 0, \quad k \in K \\
& \sum_{i \in I} x_i^k \leq 1 \\
& x_i^k \geq 0
\end{aligned}
$$

PPP: MILP formulation

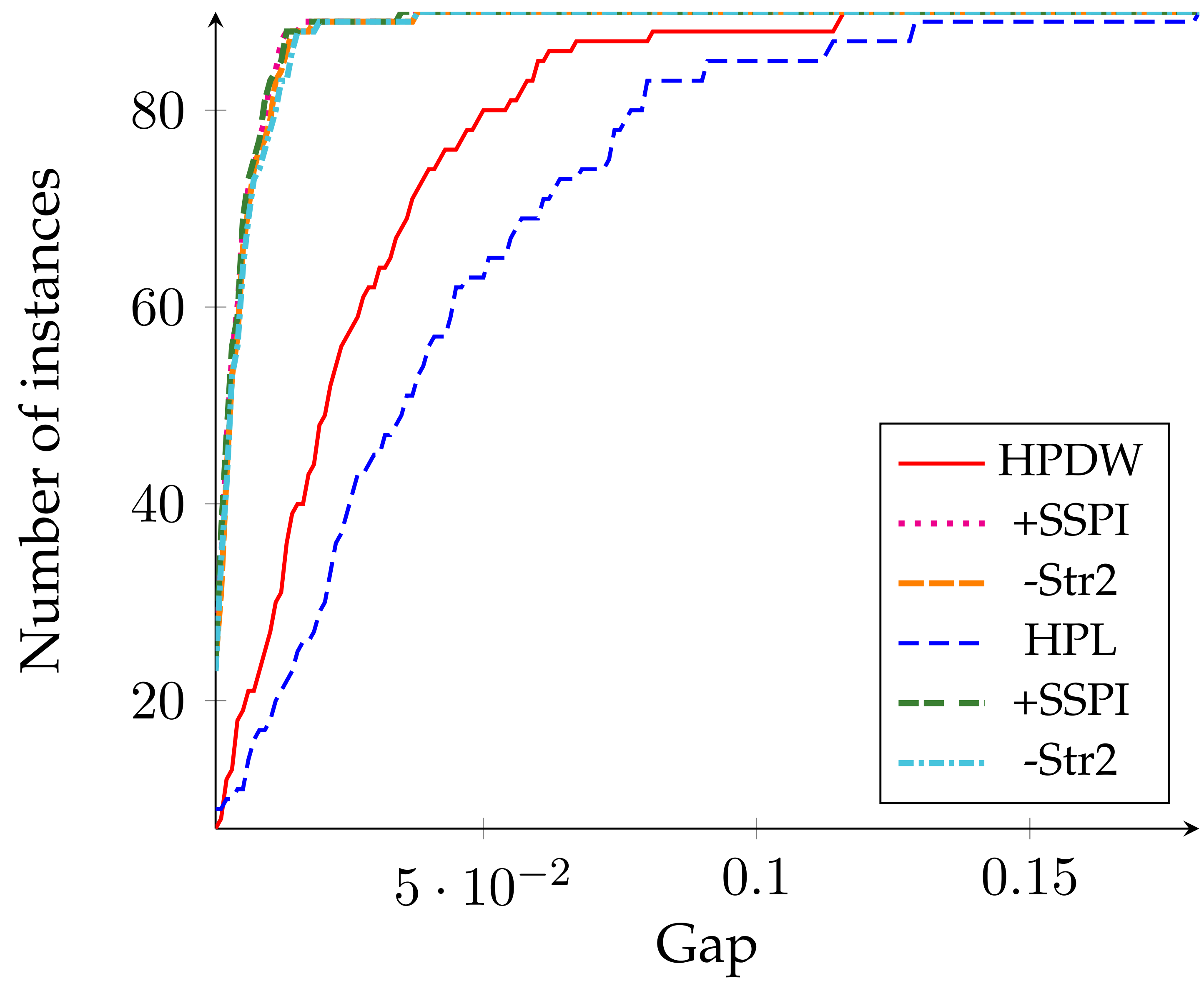
(Heilporn et al., 2010, 2011)

- MILP formulations
- Convex hull for k=1
- Branch & cut, branch & price

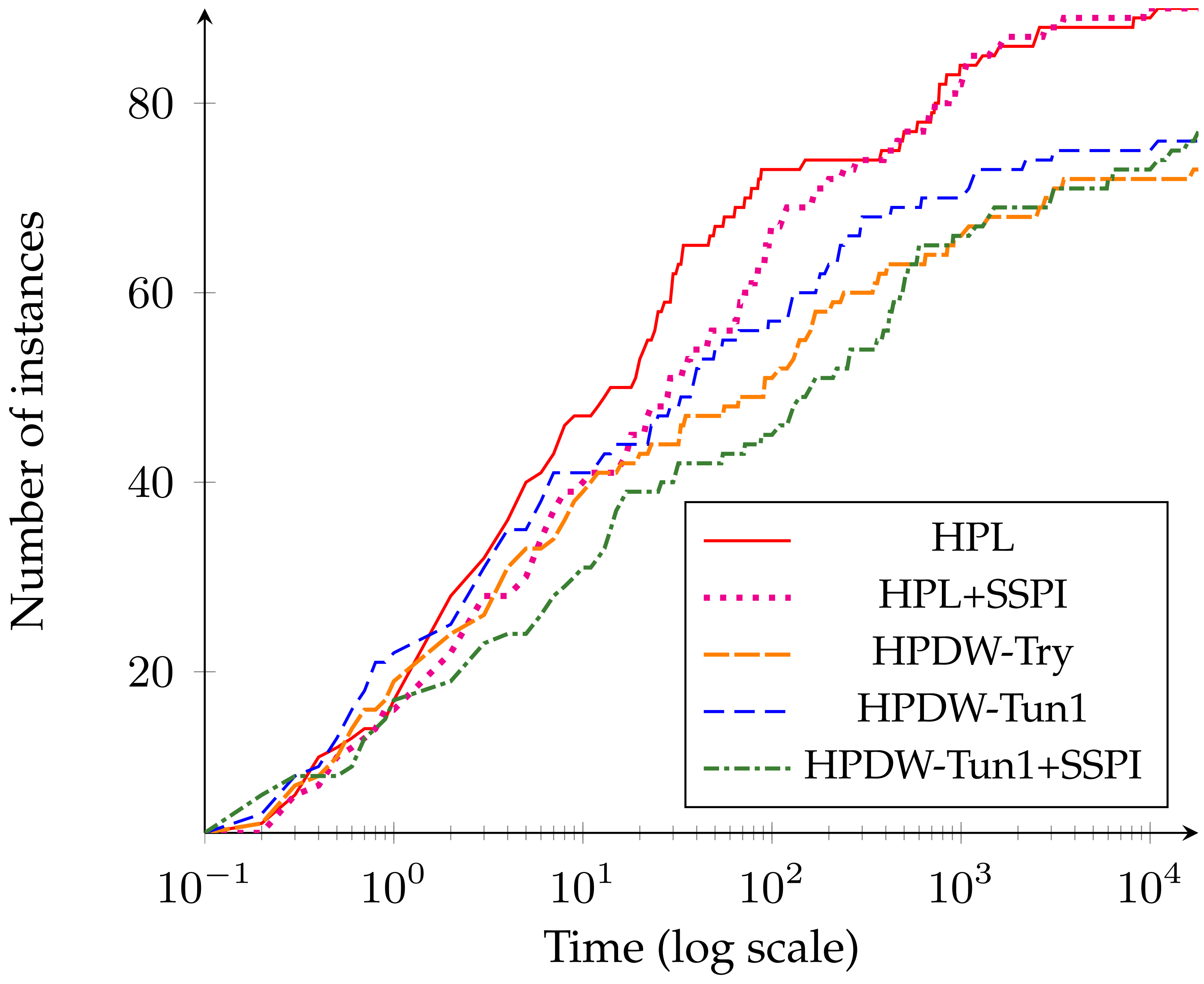
PPP: gap

(Violin, 2014)

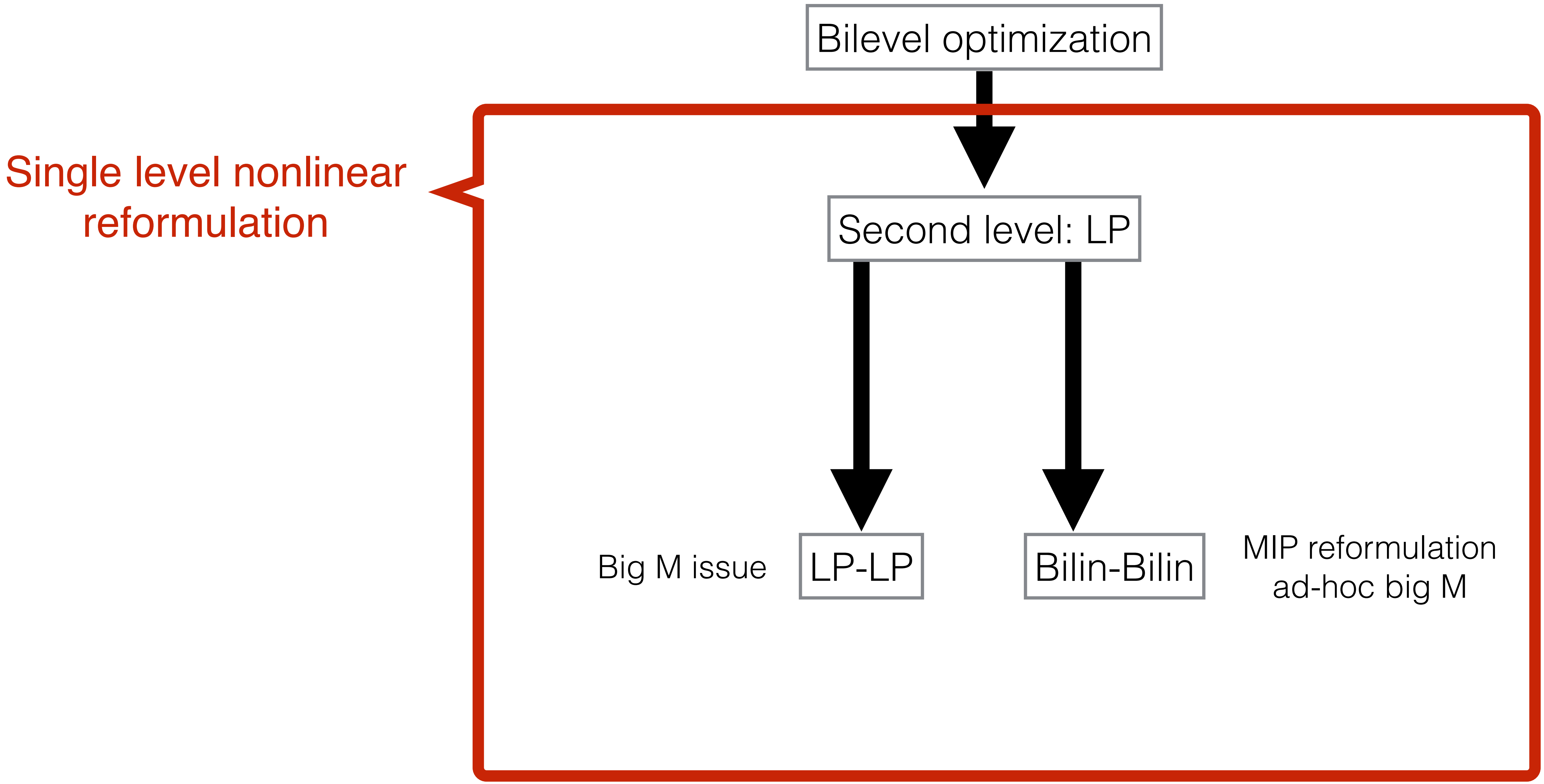
20 - 90 products
20 - 90 customers



PPP: computing time



RECAP


Bilevel optimization
Single level nonlinear reformulation
Second level: LP
Big M issue
LP-LP
Bilin-Bilin
MIP reformulation
ad-hoc big M


THANK YOU


Questions?