Complementarity Modeling of a Ramsey-Type Equilibrium Problem with Heterogeneous Agents

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Abstract. We contribute to the field of Ramsey-type equilibrium models with heterogeneous agents. To this end, we state such a model in a time-continuous and time-discrete form, which in the latter case leads to a finite-dimensional mixed complementarity problem. We prove the existence of solutions of the latter problem using the theory of variational inequalities and present further properties of its solutions. Finally, we apply our model in a case-study to real-world data to highlight the main effects of equilibria between heterogeneous households.

1. Introduction

This paper copes with a long-lasting and unresolved issue in applied economics—namely analyzing an economy with many heterogeneous agents within a fully fledged general equilibrium framework. From a theoretical point of view, the results by Mantel, Sonnenschein, and Debreu show that such models suffer from the *anything-goes* property; see, e.g., [13] or Chapter 17.E in [14] for an overview. Assumptions imposed on the micro-economic level on preferences (such as completeness, strict convexity, monotonicity, continuity; see, e.g., [6]) and technology (such as that they exhibit constant returns to scale) do not carry over to interesting properties of the competitive market equilibrium. It is not even possible to prove the most basic economic intuitions such as the law of demand; see, e.g., [17]. Reversing this reasoning means that any economic outcome—reasonable or not—may emerge from a well-behaved micro-economic basis. Boldrin and Montrucchio have shown in [4] that a similar problem arises in the neo-classical growth model (see, e.g., [16]): Any growth path can be engineered by an appropriate choice of utility functions.

An implication of the *anything-goes* theorem is that most growth models do not allow for heterogeneity among agents. Instead, the so-called representative agent or a benevolent dictator enters the scene. By construction, this excludes the analysis of distributional issues. We take up this heterogeneity issue again in this paper. To this end, we model a Ramsey-type growth process with heterogeneous agents. The resulting equilibrium model is obtained by the optimization problems of the heterogeneous households as well as those of the production sector, which are coupled using suitably chosen equilibrating conditions for interest and wage rates. The equilibrium model in time-continuous and discretized form is presented in Section 2 as a mixed complementarity problem (MCP). The relation between MCPs and variational inequalities (VIs) is then used in Section 3 to prove existence of equilibria by exploiting the classical theory of VIs. Finally, in Section 4, we apply our modeling in a case study to real-world data. The paper ends with some concluding remarks in Section 5.

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Table 1. Functions and constants used in the model

Symbol	Explanation	Range
$c_i(t)$	Consumption	$\mathbb{R}_{\geq 0}$
$a_i(t)$	Capital asset	$\mathbb{R}_{\geq 0}$
$u_i(t)$	Utility function	$\mathbb{R}_{\geq 0}$
γ_i	Utility discount rate	$(0,1)$
δ	Depreciation rate	$(0,1)$
$l_i(t)$	Labor	$\mathbb{R}_{\geq 0}$
$w(t)$	Wage rate	$\mathbb{R}_{\geq 0}$
$r(t)$	Interest rate	$\mathbb{R}_{\geq 0}$
$K(t)$	Aggregated capital	$\mathbb{R}_{\geq 0}$
$L(t)$	Aggregated labor	$\mathbb{R}_{\geq 0}$
$\mathcal{A}(t)$	Exogenous productivity factor	$\mathbb{R}_{\geq 0}$
$F(\mathcal{A},K,L)$	Production function	$\mathbb{R}_{\geq 0}$

2. Continuous Modeling and Discretization

2.1. **A Time-Continuous Model.** We consider an equilibrium version of a Ramsey-type growth model with households $i \in \mathcal{H} = \{1,\dots,I\}$. First, we are interested in a time-continuous Ramsey model. Thus, we consider a time interval $[0,T]$ with $T \in \mathbb{R}_{\geq 0}$. In the following, $c_i(t), a_i(t) : [0,T] \to \mathbb{R}_{\geq 0}$ model consumption and capital asset of household $i \in \mathcal{H}$. The optimization problem of a household $i \in \mathcal{H}$ is given by

$$\begin{aligned} \max_{c_i(\cdot),a_i(\cdot)} \quad & \int_0^T u_i(c_i(t)) e^{-\gamma_i t} \, \mathrm{d}t \\ \text{s.t.} \quad & \frac{\mathrm{d}}{\mathrm{d}t} a_i(t) = w(t) l_i(t) + (r(t)-\delta) a_i(t) - c_i(t), \quad t \in [0,T], \\ & a_i(t) \geq 0, \quad c_i(t) \geq 0, \quad t \in [0,T], \\ & a_i(0) = a_i^0, \quad a_i(T) \geq a_i^T, \end{aligned} \tag{1}$$

where u_i is an isoelastic utility function with a positive degree of relative risk aversion, $\gamma_i \in (0,1)$ is a utility discount rate, $\delta \in (0,1)$ is the depreciation rate, a_i^0 is the initial capital asset, and a_i^T is the minimum final capital stock. In the following, we assume $\sum_{i\in\mathcal{H}} a_i^0 > 0$, $\sum_{i\in\mathcal{H}} a_i^T > 0$, that labor $l_i(t)$ has a given wage rate $w(t)$, and that capital asset $c_i(t)$ has a given interest rate $r(t)$ for all $i \in \mathcal{H}$. Moreover, labor is a strictly positive and exogenously given function, i.e., $l_i(t) > 0$ for all $t \in [0,T]$.

In what follows, we use the utility functions

$$u_i(c_i) = \begin{cases} \frac{c_i^{1-\eta_i}-1}{1-\eta_i}, & \eta_i > 0,\ \eta_i \neq 1, \\ \log(c_i), & \eta_i = 1, \end{cases}$$

for $i \in \mathcal{H}$ with preferences η_i, which is of CRRA (constant relative risk aversion) type. Note that these specific choices for u_i satisfy the following standard assumptions on utility functions, see, e.g., Chapter 8.1 in [1]: They are twice differentiable and it holds $u_i' > 0$, $u_i'' < 0$, i.e., u_i are concave and strictly increasing. Moreover, the so-called Inada conditions

$$\lim_{x\to\infty} u_i'(x) = 0 \quad \text{and} \quad \lim_{x\to 0} u_i'(x) = \infty$$

hold. Finally, for $n \in \mathbb{N}$, $a_1, \dots, a_n$, $b_1, \dots, b_n$, $\beta \in \mathbb{R}_{>0}$ with $a_i = \beta b_i$ for all $i = 1, \dots, n$ it holds

$$u_i' \left(\frac{\sum_{i=1}^n u_i'(a_i)}{\sum_{i=1}^n u_i'(b_i)} \right)^{-1} = \frac{a_j}{b_j}, \quad j = 1, \dots, n.$$

In addition to the household model above, we consider a single firm, which maximizes its profit in each point of time, i.e.,

$$\max_{K(t), L(t) \geq 0} \; F(\mathcal{A}(t), K(t), L(t)) - r(t)K(t) - w(t)L(t), \tag{2}$$

for all $t \in [0, T]$, where $\mathcal{A}(t)$ is an exogenously given productivity factor, $K(t)$ is the engaged capital at given price $r(t)$, and $L(t)$ is the engaged labor at given wage rate $w(t)$, respectively. The production function is of Cobb–Douglas type, i.e.,

$$F(\mathcal{A}(t), K(t), L(t)) = \mathcal{A}(t)K(t)^{\alpha}L(t)^{1-\alpha} \quad \text{for some} \quad \alpha \in (0, 1). \tag{3}$$

To obtain an equilibrium model, we need equilibrating conditions that the firm can use at most the households aggregated capital and at most their aggregated labor, i.e.,

$$0 \leq r(t) \perp \sum_{i \in \mathcal{H}} a_i(t) - K(t) \geq 0, \quad 0 \leq w(t) \perp \sum_{i \in \mathcal{H}} l_i(t) - L(t) \geq 0 \tag{4}$$

holds for all $t \in [0, T]$. In summary, the equilibrium problem in continuous time is given by

households (1) for all $i \in \mathcal{H}$, firm (2), equilibrating conditions (4).

2.2. **Discretization.** For a time discretization of the derived equilibrium problem we assume a finite termination time $T \in \mathbb{R}_{\geq 0}$. We discretize using n intervals given by $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = T$ with interval lengths $\tau_k = t_{k+1} - t_k$ for all $k = 0, \dots, n-1$. Furthermore, we use the abbreviation $c_{i,k} = c_i(t_k)$ and the same abbreviation for the other discretized functions. Using the implicit Euler method leads to the finite-dimensional problem

$$\begin{aligned} \max_{c_i, a_i} \quad & \sum_{k=1}^n u_i(c_{i,k}) e^{-\gamma_i \sum_{m=1}^k \tau_m} \tau_k \\ \text{s.t.} \quad & \frac{1}{\tau_k}(a_{i,k+1} - a_{i,k}) = w_{k+1} l_{i,k+1} \\ & \qquad + (r_{k+1} - \delta) a_{i,k+1} - c_{i,k+1}, \quad k = 0, \dots, n-1, \\ & c_{i,k} \geq 0, \quad k = 1, \dots, n, \\ & a_{i,0} = a_i^0, \quad a_{i,n} \geq a_i^T, \quad a_{i,k} \geq 0, \quad k = 1, \dots, n-1, \end{aligned} \tag{5}$$

for every household $i \in \mathcal{H}$. Here, c_i and a_i denote the vectors of all consumption and asset variables of household i. The Karush–Kuhn–Tucker (KKT) conditions of (5), already written in MCP form, are given by

$$0 \leq -u_i'(c_{i,k}) e^{-\gamma_i \sum_{m=1}^k \tau_m} \tau_k + \lambda_{i,k-1} \perp c_{i,k} \geq 0, \quad k = 1, \dots, n, \tag{6a}$$

$$0 = a_{i,0} - a_i^0 \perp \frac{\lambda_{i,0}}{\tau_0} \text{ free}, \tag{6b}$$

$$0 \leq \lambda_{i,k-1} \left(\frac{1}{\tau_{k-1}} - (r_k - \delta) \right) - \frac{\lambda_{i,k}}{\tau_k} \perp a_{i,k} \geq 0, \quad k = 1, \dots, n-1, \tag{6c}$$

$$0 \leq \lambda_{i,n-1} \left(\frac{1}{\tau_{n-1}} - (r_n - \delta) \right) \perp a_{i,n} - a_i^T \geq 0, \tag{6d}$$

$$0 = \frac{a_{i,k+1} - a_{i,k}}{\tau_k} - w_{k+1} l_{i,k+1}$$

$$-(r_{k+1} - \delta) a_{i,k+1} + c_{i,k+1} \perp \lambda_{i,k} \text{ free}, \quad k = 0, \dots, n-1 \tag{6e}$$

for all $i \in \mathcal{H}$. Note that these conditions are both necessary and sufficient in our setting.

The firm's discretized optimization problem reads

$$\max_{K_k, L_k \geq 0} \quad F(\mathcal{A}_k, K_k, L_k) - r_k K_k - w_k L_k, \quad k = 1, \dots, n. \tag{7}$$

Note that the baseline time period $k = 0$ could, in principle, also be added to replicate the benchmark parameters (see Section 4.1), which means that the firm absorbs K_0 and L_0 for the given initial interest rate r_0 and wage rate w_0 such that the baseline period data for K and L result from our model. However, for the ease of presentation, we omit this index in the following. The (again necessary and sufficient) KKT conditions of (7) in MCP form are given by

$$\begin{aligned} 0 &\leq -F'_K(\mathcal{A}_k, K_k, L_k) + r_k \perp K_k \geq 0, \quad k = 1, \dots, n, \\ 0 &\leq -F'_L(\mathcal{A}_k, K_k, L_k) + w_k \perp L_k \geq 0, \quad k = 1, \dots, n, \end{aligned} \tag{8}$$

where κ and ϕ are the dual variables of the inequality constraints in (7).

The discretized equilibrating conditions read

$$0 \leq r_k \perp \sum_{i \in \mathcal{H}} a_{i,k} - K_k \geq 0, \quad 0 \leq w_k \perp \sum_{i \in \mathcal{H}} l_{i,k} - L_k \geq 0, \quad k = 1, \dots, n. \tag{9}$$

Putting everything together, the discretized Ramsey-like equilibrium problem is to find a solution of the MCP

$$\text{Households (6) for all } i \in \mathcal{H}, \quad \text{firm (8)}, \quad \text{equilibrating conditions (9)}. \tag{10}$$

For the following section we finally need to discuss the domains of the parameters K_k and L_k of the Cobb–Douglas production function in (7). The partial derivatives are given by

$$F'_{K_k}(\mathcal{A}_k, K_k, L_k) = \frac{\alpha}{K_k} F(\mathcal{A}_k, K_k, L_k), \quad F'_{L_k}(\mathcal{A}_k, K_k, L_k) = \frac{1-\alpha}{L_k} F(\mathcal{A}_k, K_k, L_k).$$

To ensure that the KKT conditions of (7) are well-defined and that production cannot reach infinity, we make the following standard assumption.

Assumption 1. *There exist constants $m > 0$ and $M < \infty$ so that $K_k, L_k \geq m$ and $K_k, L_k \leq M$ for all k.*

3. Existence of Equilibria

In order to show existence of equilibria of (10), we use the classical theory of variational inequalities (VIs); see, e.g., [9]. To this end, we re-state (10) as the VI

$$F(x)^\top (y - x) \geq 0 \quad \text{for all} \quad y \in X \tag{11}$$

with

$$\begin{aligned} X &= \mathbb{R}^{|\mathcal{H}|n}_{\geq 0} \times X_2 \times \mathbb{R}^{|\mathcal{H}|n} \times \mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{\geq 0} \times \mathbb{R}^n_{\geq 0}, \\ X_2 &= \prod_{i \in \mathcal{H}} \left(\{a_i^0\} \times \mathbb{R}^{n-1}_{\geq 0} \times \mathbb{R}_{\geq a_i^T} \right), \\ F(x) &= (F_j(x))_{j=1}^7 \text{ with } x = (c^\top, a^\top, \lambda^\top, K^\top, L^\top, r^\top, w^\top)^\top \end{aligned}$$

and

$$F_1(x) = \left(-u_i'(c_{i,k}) e^{-\gamma \sum_{m=1}^k \tau_m} \tau_k + \lambda_{i,k-1} \right)_{k=1,\dots,n,\, i \in \mathcal{H}},$$

$$F_2(x) = \left(\begin{pmatrix} \frac{\lambda_{i,0}}{\tau_0} \\ \left(\lambda_{i,k-1}\left(\frac{1}{\tau_{k-1}} - (r_k - \delta)\right) - \frac{\lambda_{i,k}}{\tau_k}\right)_{k=1,\ldots,n-1} \\ \lambda_{i,n-1}\left(\frac{1}{\tau_{n-1}} - (r_n - \delta)\right) \end{pmatrix} \right)_{i\in\mathcal{H}},$$

$$F_3(x) = \left(\frac{a_{i,k+1} - a_{i,k}}{\tau_k} - w_{k+1}l_{i,k+1} - (r_{k+1} - \delta)a_{i,k+1} + c_{i,k+1}\right)_{k=0,\ldots,n-1,\, i\in\mathcal{H}},$$

$$F_4(x) = \left(-F_K'(\mathcal{A}_k, K_k, L_k) + r_k\right)_{k=1,\ldots,n},$$

$$F_5(x) = \left(-F_L'(\mathcal{A}_k, K_k, L_k) + w_k\right)_{k=1,\ldots,n},$$

$$F_6(x) = \left(\sum_{i\in\mathcal{H}} a_{i,k} - K_k\right)_{k=1,\ldots,n}, \quad F_7(x) = \left(\sum_{i\in\mathcal{H}} l_{i,k} - L_k\right)_{k=1,\ldots,n}.$$

It is easy to see that the Jacobian of F is not symmetric on X. For instance, $\frac{\mathrm{d}}{\mathrm{d}K_k}F_6(x)_k = -1 \neq 1 = \frac{\mathrm{d}}{\mathrm{d}r_k}F_4(x)_k$ holds. Thus, there is no function f with $\nabla f = F$, i.e., it is not possible to solve an optimization problem for solving the $\mathrm{VI}(F, X)$; see, e.g., Theorem 1.3.1 in [9].

We now first collect some general properties of the solutions of the VI with the overall goal to prove the existence of equilibria. First, we show that every household is consuming at every point in time.

Proposition 1. *Suppose that x^* is a solution of the VI (11). Then, $c_{i,k}^* > 0$ holds for all $i \in \mathcal{H}$ and all $k \in \{1, \ldots, n\}$.*

Proof. Assume there exists $i \in \mathcal{H}$ and $k \in \{1, \ldots, n\}$ with $c_{i,k}^* \geq 0$ being arbitrarily small. The choice of the utility function implies $u_i'(c) \to \infty$ for $c \to 0$. Since x^* is a solution, we have $F_1(x^*) \geq 0$, yielding that $\lambda_{i,k}^*$ is getting arbitrarily large for $c_{i,k}^*$ getting arbitrarily small. Thus, $\lambda_{i,k}^*$ would be unbounded, which cannot be a solution of the VI. □

The next proposition states that for every point in time except for the last one, there is at least one household with strictly positive asset.

Proposition 2. *Suppose that Assumption 1 holds and that x^* is a solution of the VI (11). Then, for each $k \in \{1, \ldots, n-1\}$, there exists a household $i \in \mathcal{H}$ with $a_{i,k}^* > 0$.*

Proof. Assumption 1 implies $F_4(x^*) = 0$ by complementarity. Because K_k^* and L_k^* are bounded above as well as bounded away by a constant from 0 and since F' is continuous, we have that r_k^* is bounded in the same way. Hence, $F_6(x) = 0$ holds by complementarity and $0 < K_k^* = \sum_{i\in\mathcal{H}} a_{i,k}^*$ holds, which proves the proposition. □

Note that our numerical results in Section 4 show that there indeed are households with zero asset for some time periods.

By reasons of optimality, it is expected that the asset's lower bound at the end of the time horizon is binding. However, this is only the case under certain assumptions on the discretization of the MCP, which leads to an a-priori criterion for the final time discretization being reasonable.

Proposition 3. *Suppose that Assumption 1 holds, that x^* is a solution of the VI (11), and that $\tau_{n-1} < (\bar{r} - \delta)^{-1}$ holds with $\bar{r} := F_K'(\mathcal{A}_k, \sum_{i\in\mathcal{H}} a_i^T, \sum_{i\in\mathcal{H}} l_{i,n})$. Then, $a_{i,n}^* = a_i^T$ holds for all $i \in \mathcal{H}$, i.e., the households' final asset constraint is binding.*

Proof. It holds

$$r_n^* = F_K'\left(\mathcal{A}_k, \sum_{i\in\mathcal{H}} a_{i,n}^*, \sum_{i\in\mathcal{H}} l_{i,n}\right) \leq F_K'\left(\mathcal{A}_k, \sum_{i\in\mathcal{H}} a_i^T, \sum_{i\in\mathcal{H}} l_{i,n}\right) = \bar{r},$$

since $\sum_{i\in\mathcal{H}} a_{i,n}^* \geq \sum_{i\in\mathcal{H}} a_i^T$. We prove the statement via contradiction. Hence, we assume that $a_{i,n}^* > a_i^T$ holds for a household $i \in \mathcal{H}$. From the complementarity condition it follows $\lambda_{i,n-1}^*(1/\tau_{n-1} - (r_n^* - \delta)) = 0$. Thus, either $\lambda_{i,n-1}^* = 0$ holds, leading to

$$0 = \lambda_{i,n-1}^* \geq u_i'(c_{i,n}^*)e^{-\gamma\sum_{m=1}^k \tau_m}\tau_n,$$

which contradicts the properties of the chosen utility function, or $1/\tau_{n-1} - (r_n^* - \delta) = 0$ needs to hold, which yields

$$0 = \frac{1}{\tau_{n-1}} + \delta - r_n^* > \frac{1}{\tau_{n-1}} + \delta - \bar{r}.$$

However, since τ_{n-1} is chosen so that $1/\tau_{n-1} + \delta - \bar{r} > 0$ holds, we also obtain a contradiction in this case as well. □

In the numerical results discussed in Section 4 we thus choose an equidistant stepsize and ensure that this stepsize satisfies the condition in Proposition 3.

Next, we show an aggregation theorem that relates the VI for multiple but homogeneous households to a VI for a single but properly chosen household.

Theorem 1. *Suppose that Assumption 1 holds and that τ_{n-1} is chosen such that Proposition 3 holds. Let $x^* = (c^*, a^*, \lambda^*, K^*, L^*, r^*, w^*)$ be a solution of the VI(F, X) in (11) with $|\mathcal{H}|$ households, initial capital stocks a_i^0, and minimum final capital stocks a_i^T so that $a_i^0 = \beta a_i^T$ holds for all $i \in \mathcal{H}$ and some $\beta \in \mathbb{R}_{>0}$, i.e., the capital distribution at time 0 is the same as at time T. Furthermore, let labor $l_{i,k}$, the utility function $u_i = u$, and the discount rate $\gamma_i = \gamma$ be given. Finally, suppose that $a_{i,k}^* > 0$ holds for all $k \in \{1, \dots, n-1\}$ and $i \in \mathcal{H}$. Then, for $\tilde{\mathcal{H}}$, $|\tilde{\mathcal{H}}| = 1$, the vector $\tilde{x}^* = (\tilde{c}^*, \tilde{a}^*, \tilde{\lambda}^*, K^*, L^*, r^*, w^*)$, with*

$$\tilde{c}_k^* = \sum_{i\in\mathcal{H}} c_{i,k}^*, \quad \tilde{a}_k^* = \sum_{i\in\mathcal{H}} a_{i,k}^*, \quad \tilde{\lambda}_k^* = u'\left(\sum_{i\in\mathcal{H}} u'(\lambda_{i,k}^*)^{-1}\right),$$

is a solution of the single-household VI$(\tilde{F}, \tilde{X})$ with initial capital stock $\tilde{a}^0 = \sum_{i\in\mathcal{H}} a_i^0$, minimum final capital stock $\tilde{a}^n = \sum_{i\in\mathcal{H}} a_i^T$, labor $\tilde{l}_k = \sum_{i\in\mathcal{H}} l_{i,k}$, as well as with the same utility function u and discount rate γ as before.

The VIs depend on the initial parameters and the number of households. Thus, we denote with VI(F, X) the VI of the multi-household problem and with VI$(\tilde{F}, \tilde{X})$ the one corresponding to the aggregated, i.e., single-household, problem. Moreover, we omitted transposition of vectors for better reading.

Proof. Since x^* solves VI(F, X), it holds $x^* \in X$. From the choice of $\tilde{c}^*$, $\tilde{a}^*$, and $\tilde{\lambda}^*$ it follows $\tilde{x}^* \in \tilde{X}$. We need to show that $\tilde{x}^*$ solves VI$(\tilde{F}, \tilde{X})$. From Proposition 1 it follows $c^* > 0$, hence $F_1(x^*) = 0$ holds due to complementarity. Next, we conclude

$$\begin{aligned}\tilde{\lambda}^*_{k-1} &= u'\left(\sum_{i\in\mathcal{H}} u'(\lambda^*_{i,k-1})^{-1}\right) = u'\left(\sum_{i\in\mathcal{H}} u'\left(u'(c^*_{i,k})e^{-\gamma\sum_{m=0}^{k}\tau_m}\tau_k\right)^{-1}\right)\\ &= u'\left(\sum_{i\in\mathcal{H}} u'\left(u'(c^*_{i,k})\right)^{-1} u'\left(e^{-\gamma\sum_{m=0}^{k}\tau_m}\tau_k\right)^{-1}\right)\\ &= u'\left(\sum_{i\in\mathcal{H}} c^*_{i,k}\right) e^{-\gamma\sum_{m=0}^{k}\tau_m}\tau_k,\end{aligned}$$

by exploiting that CRRA utility functions satisfy $u'(gh) = u'(g)u'(h)$ and $u'(gh)^{-1} = u'(g)^{-1}u'(h)^{-1}$ for $g, h \in \mathbb{R}_{>0}$. This shows $\tilde{F}_1(\tilde{x}^*) = 0$.

Since $\tilde{a}^* > 0$, we need to show that $\tilde{F}_2(\tilde{x}^*)_k = 0$ holds for all $k \in \{1, \ldots, n-1\}$. Moreover, $c^* > 0$, $\tilde{c}^* > 0$, $F_1(x^*) = 0$, and $\tilde{F}_1(\tilde{x}^*) = 0$ imply $\lambda^* > 0$ and $\tilde{\lambda}^* > 0$. Exploiting that $F_2(x^*)_{i,k} = 0$ holds for all $k \in \{1, \ldots, n-1\}$ and $i \in \mathcal{H}$ yields that $\lambda^*_{i,k}/\lambda^*_{i,k-1} = R_k$ is the same for all $i \in \mathcal{H}$ by using that all households have the same utility function and time discount factor. Next, we conclude that

$$\frac{\tilde{\lambda}^*_k}{\tilde{\lambda}^*_{k-1}} = \frac{u'\left(\sum_{i\in\mathcal{H}} u'(\lambda^*_{i,k})^{-1}\right)}{u'\left(\sum_{i\in\mathcal{H}} u'(\lambda^*_{i,k-1})^{-1}\right)} = u'\left(\frac{\sum_{i\in\mathcal{H}} u'(\lambda^*_{i,k})^{-1}}{\sum_{i\in\mathcal{H}} u'(\lambda^*_{i,k-1})^{-1}}\right) = R_k$$

holds for all $k \in \{1, \ldots, n-1\}$ by using the above mentioned properties of the utility function. Thus, $\tilde{F}_2(\tilde{x}^*)_k = 0$ for all $k \in \{1, \ldots, n-1\}$. Proposition 3 implies $a^*_{i,n} = a^T_i$, hence $\tilde{a}^*_n = \tilde{a}^n$, and $\tilde{F}_2(\tilde{x}^*)_n \geq 0$ since $F_2(x^*)_{i,n} \geq 0$ for all households $i \in \mathcal{H}$. Next, $\tilde{F}_3(\tilde{x}^*)_k = \sum_{i\in\mathcal{H}} F_3(x^*)_{i,k} = 0$ holds for all $k \in \{0, \ldots, n-1\}$ because of $F_3(x^*) = 0$. Moreover, $\tilde{F}_4(\tilde{x}^*) = \tilde{F}_5(\tilde{x}^*) = 0$ is implied by $F_4(x^*) = F_5(x^*) = 0$. Finally, $\tilde{F}_6(\tilde{x}^*) = \tilde{F}_7(\tilde{x}^*) = 0$ holds because of $F_6(x^*) = F_7(x^*) = 0$, Assumption 1, $\tilde{a}^*_k = \sum_{i\in\mathcal{H}} a^*_{i,k}$, and $\tilde{l}_k = \sum_{i\in\mathcal{H}} l_{i,k}$. □

This aggregation theorem shows that Gorman's aggregation theorem, see, e.g., [1], also holds for our model applied to homogeneous households. This allows us to compare our model later on with a standard numerical approach of solving Ramsey-like growth models.

Next, we prove the existence of solution by exploiting the following classical existence result for VIs.

Theorem 2. [9, Corollary 2.2.5] *Let $X \subseteq \mathbb{R}^n$ be a nonempty, convex, and compact set and let $F\colon X \to \mathbb{R}^n$ be a continuous function. Then, the VI(X, F) has a solution.*

The VI function F is obviously continuous in our setting. However, the feasible set X is not compact but Assumption 1 can be used to show the existence of a compact and convex subset including all solutions of the original VI so that the last theorem can still be applied.

Theorem 3. *Suppose that Assumption 1 and $a^0_i \geq a^T_i$ holds as well as that $\tau_k = \tau$ and δ are chosen sufficiently small. Then, there exists a convex and compact subset $\tilde{X} \subseteq X$ such that the solutions sets of VI(X, F) and VI$(\tilde{X}, F)$ coincide.*

Proof. Let $x^* = (c^*, a^*, \lambda^*, K^*, L^*, r^*, w^*)$ be a solution of the VI(F, X).[1] Assumption 1 implies that $r^- \leq r^*_k \leq r^+$ and $w^- \leq w^*_k \leq w^+$ holds for all $k \in \{1, \ldots, n\}$ and some $r^-, r^+, w^-, w^+ \in \mathbb{R}_{>0}$. Moreover, we have $F_4(x^*) = F_5(x^*) = 0$ due to Assumption 1. Since $r^*_k > 0$ is complementary to $F_6(x^*)_k \geq 0$, and $w^*_k > 0$ to $F_7(x^*)_k \geq 0$, this implies $\sum_i a^*_{i,k} = K^*_k$ and $\sum_{i\in\mathcal{H}} l_{i,k} = L_k$. Hence,

[1] We again omit the transposition of vectors for better reading.

$a^*_{j,k} \leq \sum_{i\in\mathcal{H}} a^*_{i,k} = K^*_k$ is bounded by the upper bound of K_k for all $j \in \mathcal{H}$. We also know that $c^*_{i,k}$ cannot get arbitrarily close to 0 for all i and k. Otherwise, for some $i \in \mathcal{H}$ and for all $k \in \{1, \ldots, n\}$ we would obtain $a^*_{i,k+1} > a^*_{i,k}$ due to $F_3(x^*) = 0$, which would yield $a^*_{i,n} > a^*_{i,0}$ and, thus, a contradiction to $a^T_i \leq a^0_i$. For each household $i \in \mathcal{H}$ there is a $k \in \{0, \ldots, n-1\}$ such that $a^*_{i,k+1} \leq a^*_{i,k}$ holds. From $F_3(x^*) = 0$, it follows

$$c^*_{i,k+1} = w^*_{k+1} l_{i,k+1} + a^*_{i,k} \frac{1}{\tau_k} - a^*_{i,k+1} \left(\frac{1}{\tau_k} + \delta - r^*_{k+1} \right) \geq w^*_{k+1} l_{i,k+1} \geq w^- l_{i,k+1},$$

because $r^*_{k+1} > \delta$ is valid if δ is sufficiently small. Thus, we have a strictly positive lower bound on $c^*_{i,k+1}$. Consider now first the case $a^*_{i,k} = 0$. From $F_3(x^*) = 0$, it follows $c^*_{i,k} \geq w^*_{k+1} l_{i,k+1} \geq w^- l_{i,k+1}$. In the other case, i.e., $a^*_{i,k} > 0$, $c^*_{i,k}$ is implicitly given by

$$u' \left(\frac{c^*_{i,k+1}}{c^*_{i,k}} \right) e^{-\gamma\tau} = \tau \left(\frac{1}{\tau} - (r^*_k - \delta) \right),$$

which follows from $F_1(x^*) = 0$ and $F_2(x^*) = 0$. Hence, $c^*_{i,k}$ is bounded from below by a strictly positive constant and from above due to $F_3(x^*) = 0$ and the boundedness of a, r, and w. From $F_1(x^*) = 0$ and the continuity of u', it follows that $\lambda^*_{i,k}$ is also bounded. □

4. Numerical Results

Section 4.1 presents a real-world calibration of the model outlined in Section 2. Afterward, we discuss the computational setup and a warmstarting strategy in Section 4.2. Finally, we discuss our numerical results in Section 4.3.

4.1. **Calibration.** The benchmark data are collected from various sources; see below and [18]. They refer to Germany in 2016 as the base period. Germany's gross domestic product (GDP) was 3134 Bio. € in current prices and 38 730 € in per capita terms. The Cobb–Douglas production function (3) is calibrated mainly based on the firm's first-order optimality conditions for the base period $t = 0$, i.e., 2016. From this it follows for (3) that $\alpha = r(0)K(0)/Y(0)$. We take the off-the-shelve value $\alpha = 0.3$, GDP is normed in baseline, and we set initial production $Y(0) = 100$, gross interest $r(0) = 0.08$, and wage rate index $w(0) = 1$. Hence, we obtain $L(0) = 70$ and $K(0) = 375$. Given these numbers, $\mathcal{A}(0) = 0.8634$ holds. We assume $\mathcal{A}(t) = \mathcal{A}(0)$ to be constant over time in our case study. To induce a reasonable economic growth rate, we increase the productivity factor by 20 %, hence we adapt $\mathcal{A}(t) \leftarrow 1.2\mathcal{A}(t)$.

The CRRA-specification of instantaneous utility is often used in applied economics, e.g., in dynamic stochastic general equilibrium modeling [2] or, more generally, in monetary economics. Usually, η is referred to as the coefficient of relative risk aversion, which does not make sense in our risk-free setting. Here, η is just a measure of inter-temporal elasticity of substitution, or serves as the generational inequality aversion; see [15, p. 336]. As proposed by Nordhaus in his DICE-13 model, we use $\eta = 1.45$ and set the discount factor γ to be 0.03.

We consider $|\mathcal{H}| = 10$ households indexed by $i = 1, \ldots, 10$. They share the same endowment of labor $l_i(t) = l(t) = 7$ for all $t \in [0, T]$, $i \in \mathcal{H}$, but differ in capital asset holdings. As a proxy of initial asset holdings, we take the mean value of net wealth holdings as reported in [7, p. 31]; see Table 2. Finally, a lower bound on the terminal capital stock a^T_i of 5 % of the initial capital stock is used.

TABLE 2. Initial asset holding distribution for $|\mathcal{H}| = 10$

i	1	2	3	4	5	6	7	8	9	10
1000 €	1.382	476	258	258	99	99	19	19	10	10
share in %	62	21	6	6	2	2	0.4	0.4	0.1	0.1
$a_i(0)$	231	80	22	20	9	8	1.7	1.5	1	0.8

4.2. **Numerical Setup and Warmstart Strategy.** The numerical experiments have been carried out on a compute cluster with 755 GiB of memory and with an Intel(R) Xeon(R) CPU E5-2699 CPU. The operating system is Ubuntu 18.04.4. The instances are created by implementing Problem (10) as an MCP in GAMS 28.2.0 and are solved using PATH (version 5.0.00; see [8]) with its default settings except for the parameter `convergence_tolerance`, which is set to 10^{-5}.

To solve the instances more effectively, we use a grid refinement and warmstarting procedure in which we solve the problem with j discretization intervals and, for the next step, increase the number of intervals to $2j$. For solving the new problem, we use the solution of the coarser problem as the initial point for the problem on the finer grid. Moreover, we use mean values for the new grid points between two old ones. We repeat this procedure until we reach the required grid size. For the initialization of the first problem to be solved we use

$$a_{i,k} = a_i^0, \quad K_k = \sum_{i \in \mathcal{H}} a_{i,0}, \quad L_k = |\mathcal{H}| l_k = 70,$$
$$r_k = F_K'(\mathcal{A}(0), K_0, L_0) = 0.096, \quad w_k = F_L'(\mathcal{A}(0), K_0, L_0) = 1.2,$$
$$c_{i,k} = w_k l_{i,k} + (r_k - \delta) a_{i,k} = 8.4 + 0.046 a_i^0,$$

for $k = 0, \dots, n$ and

$$\lambda_{i,k} = -u_i'(c_{i,1}) e^{-\gamma \sum_{m=1}^{k+1} \tau_m} \tau_k,$$

for $k = 0, \dots, n-1$, which is the steady state that arises if $a_{i,k} = a_i^0$ is set for all $i \in \mathcal{H}$.

In our numerical experiments, the sketched grid refinement procedure leads to a significant speed-up. This is due to the fact that after the refinement step, the initial point generated for the next problem is of very good quality. Therefore, each refinement step takes the solver only a very small amount of iterations to converge. In contrast, solving the problem on the final grid from scratch takes rather long because the initial guess might be far away from being a solution. As a benchmark, we compare a problem with 10 households and 2000 discretization intervals. The MCP has about 68 000 rows and columns, about 240 000 non-zero entries, and takes 110 s to be solved. Compared to this, starting with 250 intervals results in a first MCP with about 8500 rows and columns as well as about 30 000 non-zero entries. We refine the grid as stated above until we reach the final number of 2000 intervals. Here, the entire solution procedure takes only 10 s, which roughly corresponds to a speed-up factor of 11. Further tests confirmed this superior performance, which is why we use the grid refinement procedure for computing all numerical results discussed in the next section. Furthermore, we have chosen the number of equidistant grid points such that Proposition 3 is fulfilled in the final refinement step, i.e., the final capital stock constraints are binding.

4.3. **Numerical Results and Economic Discussion.** The simulations discussed first in this section test and double-check the feasibility of our computational approach. Afterward, we run computational experiments on fully specified and calibrated Ramsey models with heterogeneous agents as outlined before in this

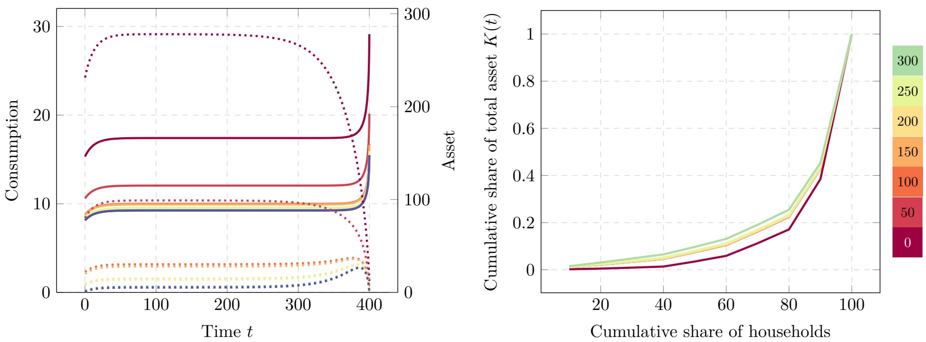

FIGURE 1. Left: Assets a_i (dotted) and consumption c_i (solid). Each color represents a different household. Right: Lorenz curves for $t \in \{0, 50, 100, 150, 200, 250, 300\}$.

paper. In our first simulation, the setup is almost exactly the same as the one discussed in [19]: agents differ only in their initial endowment with assets. The authors of [19] show that inequality in asset holdings can both increase or decrease over time, depending on the chosen parameters.[2]

As a first check, we reproduced the results from [19] to check whether both approaches yield the same numerical results. As a second check, we replicate the results from [19] and show that the inequality in terms of asset holdings decreases over time given our choice of parameters. The computational approach in [19] differs significantly from ours since it makes use of Gorman's aggregation theorem; see [12]. The theorem states that as long as welfare functions are homothetic and technologies are neoclassical, the economy can be modeled as if it were represented by a single agent. The MCP approach as outlined in this paper does not make use of this aggregation theorem. Hence, the approach is more flexible and can be applied to a much broader field of settings. However, to compare our results with those from [19], we first stick to their assumptions so that the aggregated approach is accessible. Since this allows to solve the Ramsey model as a single nonlinear optimization problem (NLP), we call it the "NLP approach" in what follows. We now consider the first check mentioned above and compare the results of the MCP approach with those for the NLP approach for $T = 400$ years. The tests verified that our results are consistent with the ones presented in [19].

Figure 1 (left) shows the consumption and asset of all 10 households and Figure 2 (solid red line) shows the firm's capital as a result of the MCP approach. In both figures, the turnpike phenomenon is clearly visible. Moreover, we see that for the period in time during which the turnpike is visible w.r.t. the firm's capital, it is also visible for the household's consumption and asset holding. During these time periods, the economy is close to a steady state; see, e.g., [1, Chapter 2].

Finally, we focus on wealth dynamics. The Lorenz curves in Figure 1 (right) display our results over time.[3] Since households are equally endowed with labor in this case study, we consider asset holdings only. One can see that the wealth

[2]Turnovsky's and Garcia-Penalosa's research is mainly inspired by the paper [5]. They assume $\delta = 0$, $T \to \infty$, and implement the more general CES production function instead of the Cobb–Douglas function. Everything else is identical to the model used here. The direction of the income distribution depends on the elasticity of substitution between labor and capital as well as on the standard deviation of relative capital.

[3]The Lorenz curve shows the proportion of wealth hold by a given proportion of agents. The proportion of agents is shown on the x-axis and the share of total assets (i.e., wealth) is shown on the y-axis. Agents are sorted in an increasing order w.r.t. asset holdings.

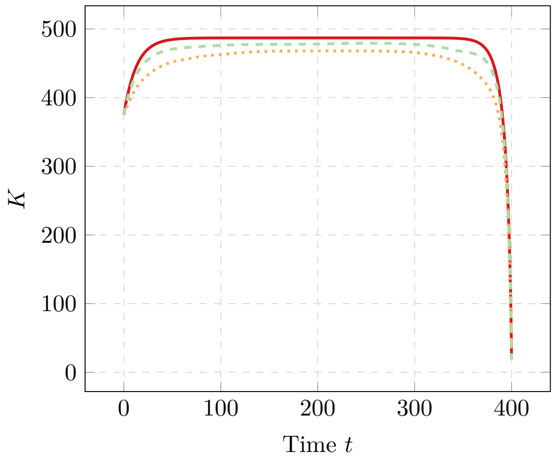

FIGURE 2. Firm's capital in the three considered cases. (i) Homogeneous households (solid red), (ii) policy maker (dashed green), (iii) different capital market access (dotted yellow).

order is preserved over time, i.e., the ranking among the agents w.r.t. wealth is preserved during the growth process. Given the model's specification and the choice of parameters, inequality decreases over time before finite time horizon effects lead to the same wealth distribution at $t = 400$ as for $t = 0$ due to our terminal capital stock constraint. This is why we are displaying the Lorenz curves for $t \in \{0, 50, 100, 150, 200, 250, 300\}$ to avoid the finite time horizon effects. Hence, economic growth leads to less inequality among agents given that they differ only in their initial endowments.

To demonstrate the versatility of the MCP approach, we now present numerical experiments, which are not accessible by traditional approaches based on aggregation as used, e.g., in [19]. First, we assume a policy maker that targets the households minimum terminal capital stock. Household $i \in \mathcal{H}$ is equipped with a minimum final capital stock condition with lower bound $a^0_{10-i}/20$, which should induce a proper dynamics in asset holdings since the ordering of the households is reverted over time. Furthermore, we heterogenize the time discount factor of each household by setting it to $0.03 + 0.001i$. Due to these modifications, the households cannot be aggregated anymore in the standard way as described above. Figure 3 (left) shows the household's consumption and asset holding. We see that due to the different final capital stock conditions and different time discount factors, the turnpike behavior is not as pronounced as in the example before. Moreover, the consumption patterns change compared to the numerical example discussed before. In Figure 3 (left) we see the turnpike behavior in consumption and asset only for those households that are running out of money. Comparing this to Figure 1 (left) shows significant changes in the turnpike pattern: Most households do not not reach a turnpike-like steady state or the time period of this state is significantly reduced. For example, household 1 (red) is increasing consumption over the time horizon. We also see some households with decreasing consumption (from $t = 340$ to $t = 380$) just before they increase it again due to considered finite time horizon. This is because the households run out of money and have to save to fulfill the final capital stock condition. The overall picture in Figure 2 (dashed green line) is still comparable to the case discussed before (solid red line) but shows a narrow bend from $t = 340$ to $t = 380$, which coincidences with the time period in which some households are lowering their consumption. The economy evolves towards a more uneven distribution of assets in the long run,

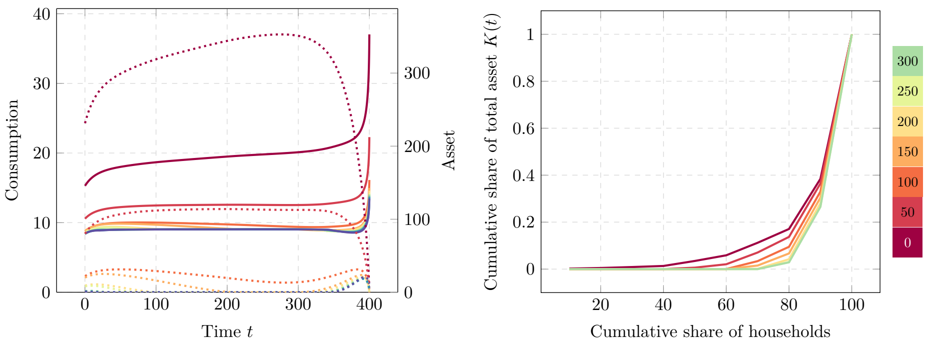

FIGURE 3. Left: Assets a_i (dotted) and consumption c_i (solid) under the assumption of a policy maker. Right: Corresponding Lorenz curves for $t \in \{0, 50, 100, 150, 200, 250, 300\}$.

hence inequality is strengthened. Figure 3 (right) shows the Lorenz curves, which also indicate increased inequality over time. Some households run out of assets completely. This is due to their higher impatience to consume as reflected by their higher discount rate. The latter, in particular, shows that even small differences in the discount rate really matter. Given their labor income, impatience to consume is too strong to overcome low interest rates on savings. This provocative result is known as the dominant consumer problem; see, e.g., [3] for an overview. It blames poor households (in terms of capital income) to be poor because of their preferences and not because of unfavorable initial capital holdings.

Another example, which clearly shows the advantage of our MCP-based modeling approach is the analysis of capital market imperfection. Access to the capital market differs among agents, e.g., because of different capacities to process information. Likewise, some agents may be subject to unfavorable taxation of their financial transactions while others can trade without transaction costs. These asymmetries are reflected in our model by assuming small differences in the rate of return. Hence, we change the household's discretized ODE by changing $r_k \leftarrow \kappa_i r_k$ for some $\kappa_i \in (0, 1]$. We choose $\kappa_i = 1.0 - 0.2i/10$ for $i \in \mathcal{H}$. Figure 4 (left) again shows the household's consumption and asset holding. Comparing these curves to the ones in Figure 1 (left) again shows a significant change in the turnpike behavior. Most households are not reaching a turnpike or only because they run out of financial assets. Comparing the firm's capital in Figure 2 (dotted yellow line) with Figure 2 (solid red line) shows a similar growth of the economy, but if we put this in context with Figure 4 (left) and Figure 1 (left), we see that the small differences in the rate of return lead to a more uneven distribution of financial assets. Household 1 is increasing its financial assets just before the finite time horizon effect applies, whereas all other households are lowering their hold of financial assets. The Lorenz curves in Figure 4 (right) illustrate this imbalance in capital distribution again. We see that the uneven distribution of capital is increasing. The implications of this asymmetry are dramatic. Only households 1 and 2 survive as "capitalists" while all other households decide to not hold financial assets in the meantime. We thus observe that imperfect capital markets strongly reinforce inequality in asset holdings.

5. CONCLUSION

In this paper, we discussed an MCP model of a time-discrete Ramsey-type equilibrium problem with heterogeneous agents, showed the existence of equilibria, and presented numerical results for a realistic calibration of the model. This paves

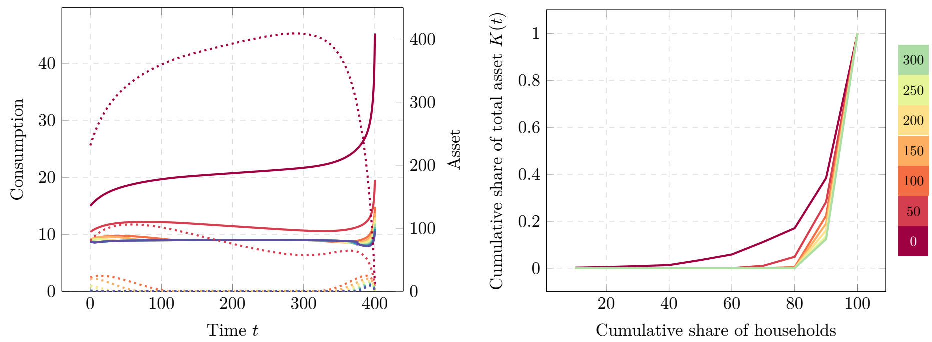

FIGURE 4. Left: Assets a_i (dotted) and consumption c_i (solid) under the assumption of better capital market access for wealthy households. Right: Corresponding Lorenz curves for $t \in \{0, 50, 100, 150, 200, 250, 300\}$

the way to consider equilibrium models of heterogeneous and additionally spatially dispersed households. In this case, spatial processes may be modeled via partial differential equations. This has been carried out in an optimal-control setting in, e.g., [10, 11], which we plan to generalize to an equilibrium setting in our future work.

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References

[1] D. Acemoglu. *Introduction to Modern Economic Growth.* Princeton University Press, 2008.

[2] S. Aruoba, J. Fernandez-Villaverde, and J. Rubio-Ramirez. "Comparing Solution Methods for Dynamic Equilibrium Economies." In: *Journal of Economic Dynamics and Control* 30 (2003), pp. 2477–2508. DOI: 10.1016/j.jedc.2005.07.008.

[3] R. A. Becker. "Equilibrium Dynamics with Many Agents." In: *Handbook on Optimal Growth 1: Discrete Time.* Ed. by R.-A. Dana, C. Le Van, T. Mitra, and K. Nishimura. Berlin, Heidelberg: Springer Berlin Heidelberg, 2006, pp. 385–442. DOI: 10.1007/3-540-32310-4_13.

[4] M. Boldrin and L. Montrucchio. "On the indeterminacy of capital accumulation paths." In: *Journal of Economic Theory* 40.1 (1986), pp. 26–39. DOI: 10.1016/0022-0531(86)90005-0.

[5] F. Caselli and J. Ventura. "A Representative Consumer Theory of Distribution." In: *American Economic Review* 90.4 (2000), pp. 909–926. DOI: 10.1257/aer.90.4.909.

[6] G. Debreu. "Excess demand functions." In: *Journal of Mathematical Economics* 1.1 (1974), pp. 15–21. DOI: 10.1016/0304-4068(74)90032-9.

[7] Deutsche Bundesbank. *Household wealth and finances in Germany: results of the 2017 survey.* Monthly Report, 2019. URL: https://www.bundesbank.de/resource/blob/796280/5eaebf3d73e4fdde961bd1dacab852f3/mL/2019-04-vermoegensbefragung-data.pdf.

[8] S. P. Dirkse and M. C. Ferris. “The path solver: a nommonotone stabilization scheme for mixed complementarity problems.” In: *Optimization Methods and Software* 5.2 (1995), pp. 123–156. DOI: 10.1080/10556789508805606.

[9] F. Facchinei and J.-S. Pang. *Finite-Dimensional Variational Inequalities and Complementarity Problems*. Springer Series in Operations Research and Financial Engineering. Springer New York, 2003. DOI: 10.1007/b97543.

[10] L. Frerick, G. Müller-Fürstenberger, E. W. Sachs, and L. Somorowsky. *A Nonlocal Spatial Ramsey Model with Endogenous Productivity Growth on Unbounded Spatial Domains*. Tech. rep. 2019. URL: https://arxiv.org/abs/1909.02348.

[11] L. Frerick, G. Müller-Fürstenberger, E. W. Sachs, and L. Somorowsky. *The Nonlocal Ramsey Model for an Interacting Economy*. Tech. rep. 2019. URL: https://arxiv.org/abs/1909.02337.

[12] W. M. Gorman. “Community Preference Fields.” In: *Econometrica* 21.1 (1953), pp. 63–80. URL: http://www.jstor.org/stable/1906943.

[13] R. Mantel. “Homothetic preferences and community excess demand functions.” In: *Journal of Economic Theory* 12 (1976), pp. 197–201. DOI: 10.1016/0022-0531(76)90073-9.

[14] A. Mas-Colell, M. D. Whinston, and J. R. Green. *Microeconomic Theory*. New York: Oxford University Press, 1995.

[15] W. Nordhaus. “Projections and Uncertainties about Climate Change in an Era of Minimal Climate Policies.” In: *American Economic Journal: Economic Policy* 10.3 (2018), pp. 333–60. DOI: 10.1257/pol.20170046.

[16] V. Quadrini and J. Ríos-Rull. “Inequality in Macroeconomics.” In: *Handbook of Income Distribution*. Handbook of Income Distribution. 2015, pp. 1229–1302. DOI: 10.1016/B978-0-444-59429-7.00015-7.

[17] J. K.-H. Quah. “The Monotonicity of Individual and Market Demand.” In: *Econometrica* 68.4 (2000), pp. 911–930. DOI: 10.1111/1468-0262.00141.

[18] Statistisches Bundesamt. *Volkswirtschaftliche Gesamtrechnungen. Inlandsproduktberechnung, Lange Reihen ab 1970*. Destatis, 2020. URL: https://www.destatis.de/DE/Themen/Wirtschaft/Volkswirtschaftliche-Gesamtrechnungen-Inlandsprodukt/Publikationen/Downloads-Inlandsprodukt/inlandsprodukt-lange-reihen-pdf-2180150.html.

[19] S. Turnovsky and C. Garcia-Penalosa. “The dynamics of wealth inequality in a simple Ramsey model: A note on the role of production flexibility.” In: *Macroeconomic Dynamics* 13 (2009), pp. 250–262. DOI: 10.1017/S1365100508070508.

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