FRIEDRICH-ALEXANDER
UNIVERSITÄT
ERLANGEN-NÜRNBERG

NATURWISSENSCHAFTLICHE
FAKULTÄT

Robust Gas Network Operation

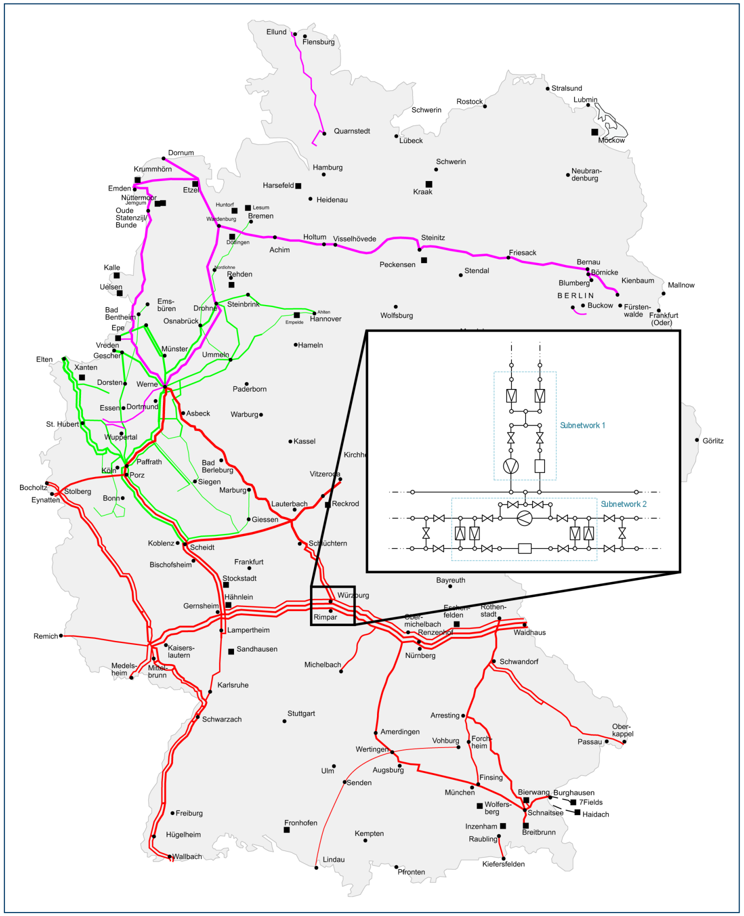
via an Outer Approximation Framework for Mixed-Integer Nonlinear Robust Optimization

D. Bernhard, M. Kuchlbauer, F. Liers, M. Stingl
FAU Erlangen-Nürnberg

OPTIMIZATION
UNDER UNCERTAINTY
& DATA ANALYSIS

Gas Networks

- large and complex
- stationary setting is non-convex
- uncertainties inherent, e.g., in demand and in physical parameters



Robust Nomination Validation in Gas Networks

Problem

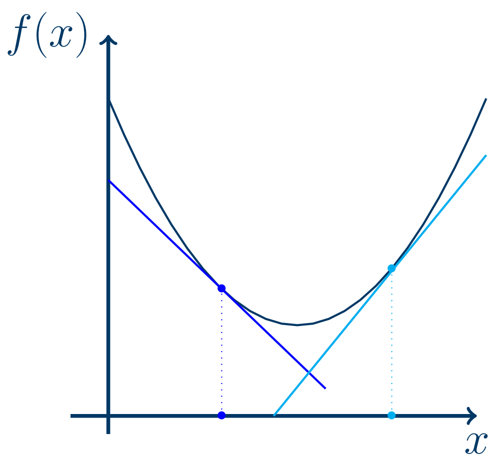
For each uncertainty, is there a configuration of the active elements leading to a feasible state?

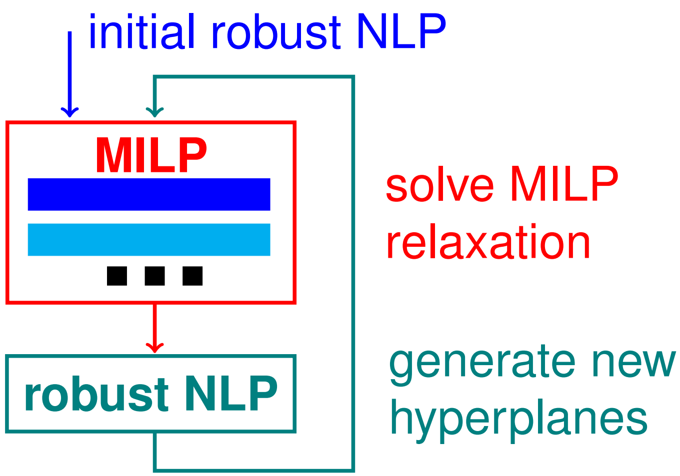
$$
\begin{aligned}
\min_{\Delta, q, \pi} \quad & c(\Delta) \\
& Aq = d && \text{(flow conservation)} \\
& (A^T \pi)_a = -\lambda_a q_a |q_a| \quad \forall a \in && \text{(pressure loss on pipes)} \\
& (A^T \pi)_a = \Delta_a \quad \forall a \in && \text{(active elements)} \\
& \pi \in [\underline{\pi}, \overline{\pi}] && \text{(pressure bounds)} \\
& q \in \mathbb{R}^{|\mathcal{A}|}.
\end{aligned}
$$

uncertainties in demands d and in physical parameters λ.

General Decomposition Approach

Discrete-Continuous Robust Optimization via Outer Approximation



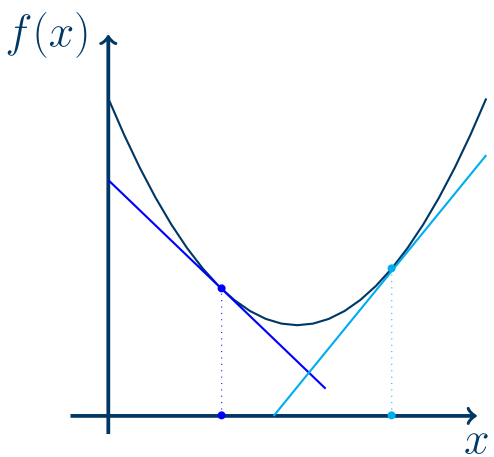


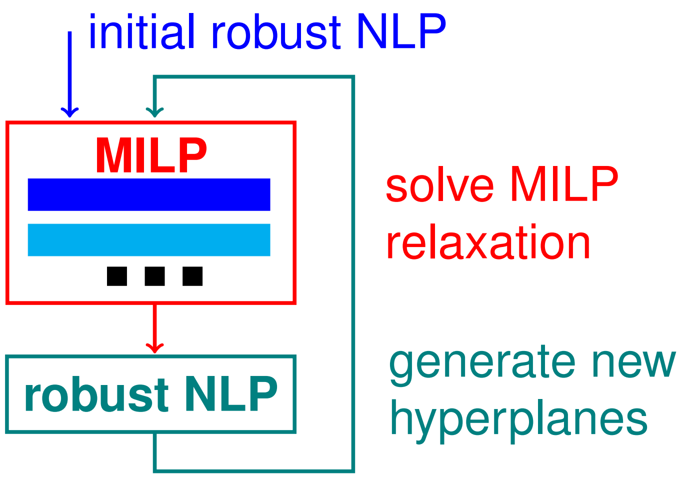
MILP $\Rightarrow$ lower bound

robust NLP $\Rightarrow$ upper bound

General Decomposition Approach

Discrete-Continuous Robust Optimization via Outer Approximation





MILP $\Rightarrow$ lower bound

robust NLP $\Rightarrow$ upper bound

- Solution of (nonlinear) robust subproblems?
  Kuchlbauer, L, Stingl (2020)
- Valid inequalities for master problems?
  Kuchlbauer, L, Stingl (2022)

Nonlinear Robust Optimization

formulation as minimax problem

$$\min_{x \in \mathbb{R}^n} \quad c(x)$$
$$v(x,u) \leq 0 \quad \forall u \in \mathcal{U}.$$

minimax problem: $$\min_{x \in \mathbb{R}^n} \max_{u \in \mathcal{U}} v(x,u). \qquad \text{(RO)}$$

challenges:

- evaluation of worst case: global solution of $\max_{u \in \mathcal{U}} v(x,u)$
- nonlinear and non-convex
- few works only (Leyffer et al., 2020), no general approaches
- known reformulations need strong structural assumptions

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
NATURWISSENSCHAFTLICHE FAKULTÄT

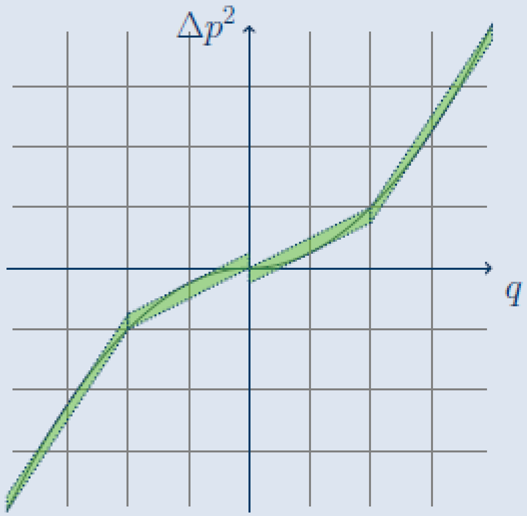
Our Solution Approach

adversarial problem

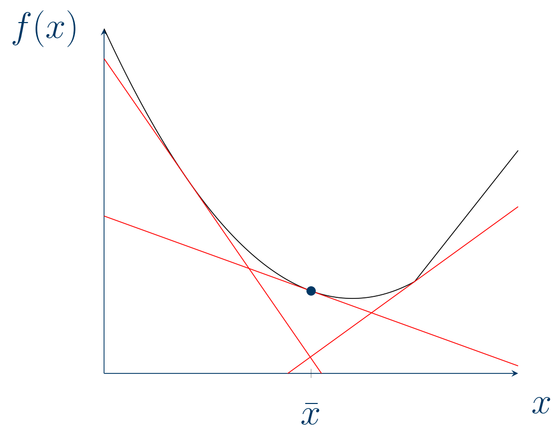
minimize the adversary's optimal value function

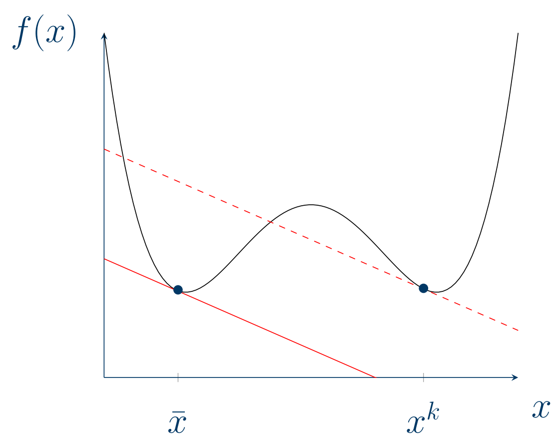
$$f(x) := \max_{u \in \mathcal{U}} v(x, u). \quad \text{(RO)}$$

- bundle method for non-smooth and non-convex function $\min_{x \in \mathbb{R}^n} f(x)$
- piecewise linear relaxation for $\max_{u \in \mathcal{U}} v(x, u)$ with guaranteed error bound Geißler et al. (2012)



Bundle Method





inner loop:

- approximate f by a piecewise linear model ϕ_k, using cutting planes
- use subgradients for nonsmooth functions
- find trial iterates around serious $\bar{x}$:

$$\min_{x^k \in \mathbb{R}^n} \phi_k(x^k) + \tau_k \|x^k - \bar{x}\|.$$

- downshift cutting planes to overcome lack of convexity

outer loop: accept trial iterate as new serious point if ϕ_k good enough

e.g., constant error in subgradient, convex: Kiwiel 2006, nonconvex: Noll 2013, Hertlein & Ulbrich (2019)

Inexactness in the Adversarial Problem

$$\min_{x \in \mathbb{R}^n} f(x) = \min_{x \in \mathbb{R}^n} \max_{u \in \mathcal{U}} v(x,u). \qquad \text{(RO)}$$

inexact solution u_x to the adversarial problem:

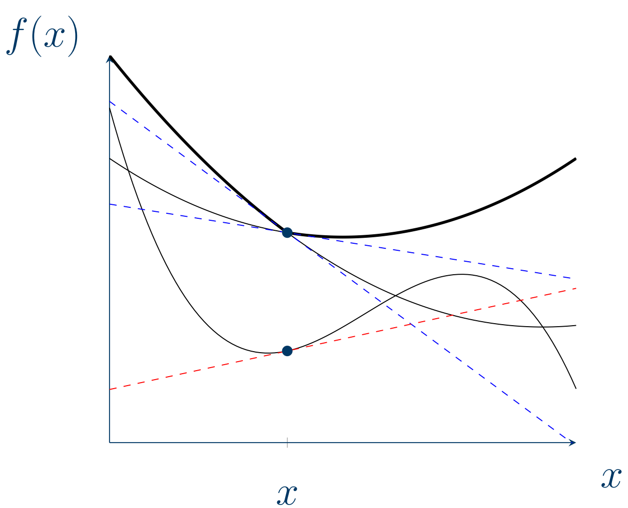
$$v(x, u_x) \geq \max_{u \in \mathcal{U}} v(x,u) - \varepsilon_x.$$

Clarke subdifferential of f at x: $\partial f(x) =$

$$\operatorname{conv}\{\partial_x v(x,u^*) \mid u^* \in \mathcal{U},\ v(x,u^*) = \max_{u \in \mathcal{U}} v(x,u)\}.$$

$$\tilde{\partial}_a f(x) := \operatorname{conv}\{\partial_x v(x,u) \mid u \in \mathcal{U},\ v(x,u) \geq v(x,u_x)\}.$$

approximate exact subdifferential from outside $\Rightarrow$ no constant error bound implied, no bundle concept available, need to generalize Noll!

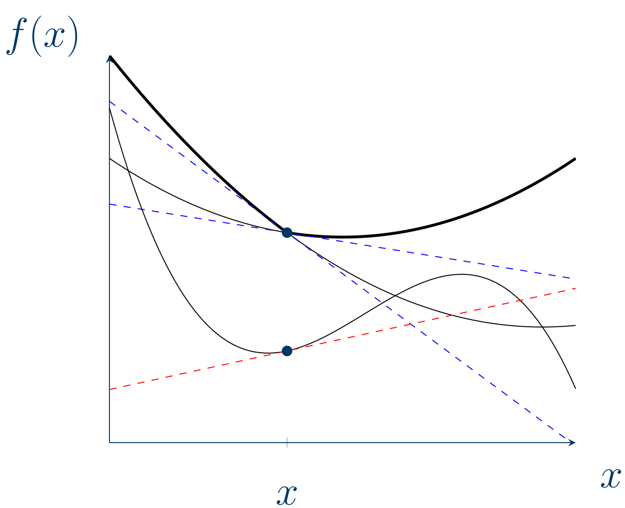


Inexactness in the Adversarial Problem

$$\min_{x\in\mathbb{R}^n} f(x) = \min_{x\in\mathbb{R}^n} \max_{u\in\mathcal{U}} v(x,u). \qquad \text{(RO)}$$

inexact solution u_x to the adversarial problem:

$$v(x,u_x) \geq \max_{u\in\mathcal{U}} v(x,u) - \varepsilon_x.$$



Clarke subdifferential of f at x: $\partial f(x) =$

$$\operatorname{conv}\{\partial_x v(x,u^*) \mid u^* \in \mathcal{U},\ v(x,u^*) = \max_{u\in\mathcal{U}} v(x,u)\}.$$

$$\tilde{\partial}_a f(x) := \operatorname{conv}\{\partial_x v(x,u) \mid u \in \mathcal{U},\ v(x,u) \geq v(x,u_x)\}.$$

approximate exact subdifferential from outside $\Rightarrow$ no constant error bound implied, no bundle concept available, need to generalize Noll!

fix by adaptivity!

Adaptive Approximation of Function Value

Definition

- $f : \mathbb{R}^n \to \mathbb{R}$ is approximate convex ($= LC^1$) if for every x and ε', there exists $\delta > 0$ s.t. f is ε'-convex on $B(x, \delta)$.
  (e.g., $\max_u f(x, u)$ with $f(., u) \in C^1 \forall u$)
- $f : \mathbb{R}^n \to \mathbb{R}$ is ε'-convex if for any $x, x' \in X$ and $\lambda \in [0, 1]$,

$$f(\lambda x + (1 - \lambda)x') \leq \lambda f(x) + (1 - \lambda) f(x') + \varepsilon' \lambda(1 - \lambda)\|x - x'\|.$$

Lemma

Under ε'-convexity, an approximate subgradient $g_k \in \tilde{\partial}_a f(x^k)$ fulfills

$$g_k{}^T(x - x^k) \leq f(x) - f(x^k) + \varepsilon_{x^k} + \varepsilon' \|x - x^k\|.$$

$\Rightarrow$ for trial iterates x^k around a serious iterate $\bar{x}$, evaluate adversary well enough: $\varepsilon_{x^k} = \varepsilon'' \|\bar{x} - x^k\|$
$\Rightarrow$ subgradient inequality $g_k{}^T(\bar{x} - x^k) \leq f_a(\bar{x}) - f_a(x^k) + (\varepsilon' + \varepsilon'')\|\bar{x} - x^k\|$.

Convergence Result

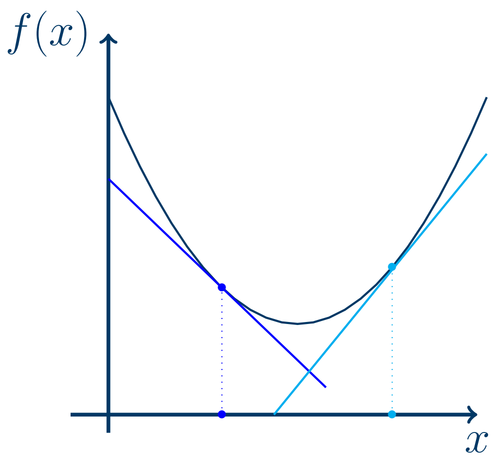
Theorem

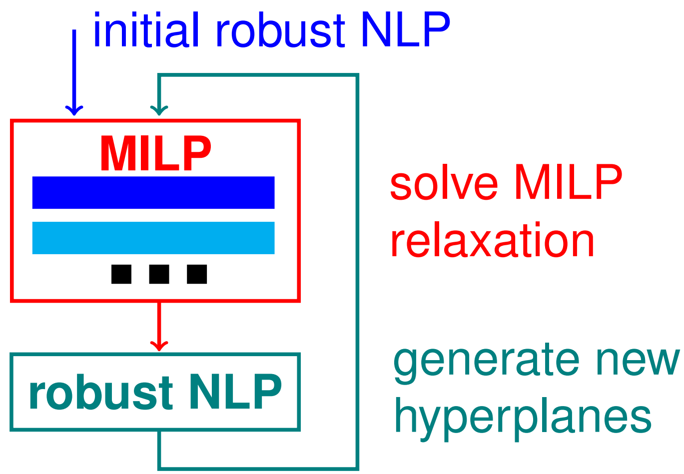
Let x_1 be s.t. $\Omega := \{x \in \mathbb{R}^n : f(x) \leq f_a(x_1)\}$ is bounded, v be approximate convex (LC^1) and $\bar{x}$ obtained by a stopping criterien or an accumulation point of serious iterates. Then, it holds that

$$0 \in \tilde{\partial}_a f(\bar{x}).$$

Optimality: $0 \in \tilde{\partial}_a f(\bar{x})$, i.e. $f(\bar{x}) \leq f(y) + \varepsilon' \|\bar{x} - y\| + \varepsilon_{\bar{x}}$, locally.

Discrete-Continuous Robust Optimization via Outer Approximation





MILP $\Rightarrow$ lower bound

robust NLP $\Rightarrow$ upper bound

- Solution of (nonlinear) robust subproblems?
  Kuchlbauer, L, Stingl (2020)
- Valid inequalities for master problems?
  Kuchlbauer, L, Stingl (2022)

FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG
NATURWISSENSCHAFTLICHE FAKULTÄT

Convex Mixed-Integer Robustness

$$\min_{x,y} \quad C(x,y)$$

$$G(x,y) := \max_{u \in \mathcal{U}} \sum_{i=1}^{n} V_i^{+}(x,y,u) \leq 0,$$

$$x \in X, y \in Y \cap \mathbb{Z}^{n_y}.$$

C convex in x, y, allow non-convex adversarial

bundle method $\rightarrow$ subproblem solutions and cutting planes

- generalizes OA proofs [Fletcher, Leyffer, 1994; Delfino, Oliveira, 2018; Wei et al. 2019]
- inexact cutting planes: valid, but acceptance of ε_k-feasible solutions

Theorem

The outer approximation method together with the adaptive bundle method terminates after finitely many outer approximation iterations and either detects infeasibility or outputs a solution that is ε_k-feasible and ε_{oa}-optimal.

Robust Nomination Validation in Gas Networks

Problem

For each uncertainty, is there a configuration of the active elements leading to a feasible state?

$$\min_{\Delta,q,\pi} \quad c(\Delta)$$

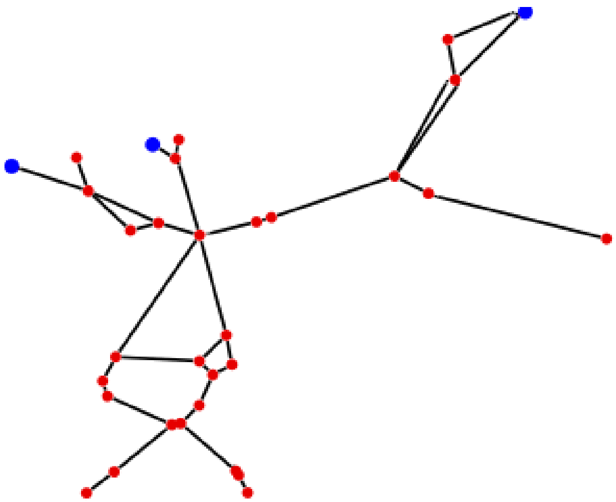
$$\begin{aligned}
& Aq = d && && \text{(flow conservation)} \\
& (A^T\pi)_a = -\lambda_a q_a |q_a| && \forall a \in \mathcal{A}_{pi} && \text{(pressure loss on pipes)} \\
& (A^T\pi)_a = \Delta_a && \forall a \in \mathcal{A}_{ac} && \text{(active elements)} \\
& \pi \in [\underline{\pi}, \overline{\pi}] && && \text{(pressure bounds)} \\
& q \in \mathbb{R}^{|\mathcal{A}|}. && &&
\end{aligned}$$

uncertainties in demands d and in physical parameters λ.
in bundle: Adversary maximizes constraint violations.

TRR 154

Numerical Results on Realistic Instances

nodes × arcs	compr.	valves	runtime	error constr.
11×11	2	1	13	0
24×25	4	0	8	0
40×45	5	2	903	0
103×105	21	3	362	5e-5



General approach can go to realistic sizes.

Curently: Robust Chance-Constrained Optimization

$$\min_{\Delta \in [\underline{\Delta}, \overline{\Delta}]} \quad C(\Delta)$$
$$s.t. \quad \mathbb{P}_{d \sim \zeta}(\pi_v(\Delta, d, \lambda) \in [\underline{\pi}_v, \overline{\pi}_v] \; \forall v \in V) \geq p \qquad \forall \zeta \in U.$$

- assume discrete, uncertain, probability distribution in ambiguity set U
- adaptive bundle method applicable for (approximate) robust joint CC $\Rightarrow$ local solution

preliminary computational results: U built from confidence intervals

- gaslib24, gaslib40, 500 scen.: ca. 170 s, 1000 scen.: ca. 300 s
- gaslib134, 500 scen.: 172 s, 1000 scen: 400 s

Curently: Robust Chance-Constrained Optimization

$$
\begin{aligned}
\min_{\Delta \in [\underline{\Delta}, \overline{\Delta}]} \quad & C(\Delta) \\
s.t. \quad & \mathbb{P}_{d \sim \zeta}(\pi_v(\Delta, d, \lambda) \in [\underline{\pi}_v, \overline{\pi}_v] \; \forall v \in V) \geq p \qquad \forall \zeta \in U.
\end{aligned}
$$

- assume discrete, uncertain, probability distribution in ambiguity set U
- adaptive bundle method applicable for (approximate) robust joint CC $\Rightarrow$ local solution

preliminary computational results: U built from confidence intervals

- gaslib24, gaslib40, 500 scen.: ca. 170 s, 1000 scen.: ca. 300 s
- gaslib134, 500 scen.: 172 s, 1000 scen: 400 s

Thanks for your attention.