A Branch-and-Bound Algorithm for Nonconvex Nash Equilibrium Problems

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This is joint work with

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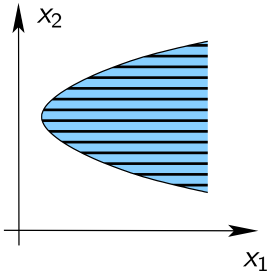
Survey

A Nash equilibrium problem (based on Beck/St. 2024)

For convex quadratic functions $q_1, q_2 : \mathbb{R}^1 \to \mathbb{R}^1$ consider the two parametric problems

$$
\begin{aligned}
P_1(x_2): &\quad \min_{x_1} x_1 \quad \text{s. t.} \quad q_1(x_2) \le x_1, \\
P_2(x_1): &\quad \min_{x_2} x_2 \quad \text{s. t.} \quad q_2(x_1) \le x_2.
\end{aligned}
$$

A Nash equilibrium problem



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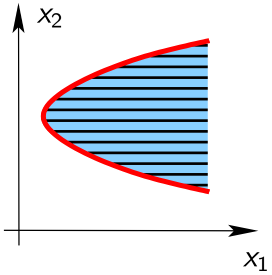

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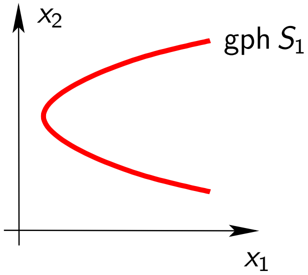

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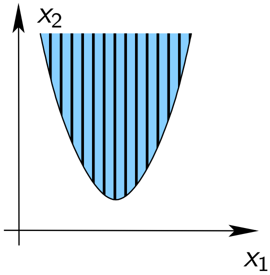

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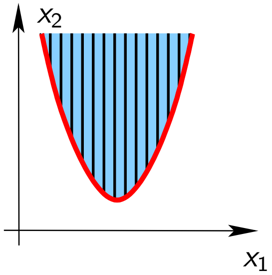

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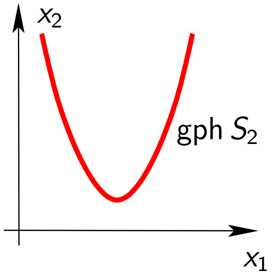

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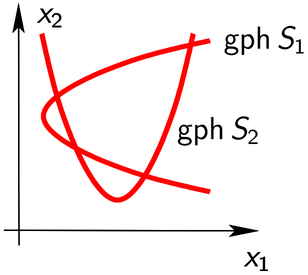

$$x^{1,\star} \in S_1(x^{2,\star}), \quad x^{2,\star} \in S_2(x^{1,\star})$$

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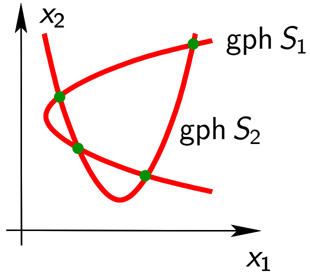

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General problem definition

We consider Nash equilibrium problems of the following form:

- Finite number of players $\nu = 1, \dots, N$.
- Strategy sets are boxes $\Omega_\nu \subseteq \mathbb{R}^{n_\nu}$.
- Each player ν's objective function

$$\theta_\nu : \Omega_1 \times \ldots \times \Omega_N \to \mathbb{R}$$

  is continuous.
- Each player ν only controls her variables $x^\nu \in \Omega_\nu$,
  but her objective function also depends on all other players' decisions (as parameters).
- Put $n = \sum_{\nu=1}^{N} n_\nu$ and $\Omega := \Omega_1 \times \ldots \times \Omega_N \subseteq \mathbb{R}^n$.

General problem definition

A Nash equilibrium is a point $\bar{x} = (\bar{x}^1, \ldots, \bar{x}^N) \in \Omega$ such that for each $\nu = 1, \ldots, N$ the point $\bar{x}^\nu$ is a global minimal point of the parametric optimization problem

$$P_\nu(\bar{x}^{-\nu}): \quad \min_{x^\nu} \ \theta_\nu(x^\nu, \bar{x}^{-\nu}) \quad \text{s. t.} \quad x^\nu \in \Omega_\nu.$$

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The collection of the problems $P_\nu(x^{-\nu})$, $\nu = 1, \ldots, N$, with $x \in \Omega$ is called Nash equilibrium problem (NEP).

Literature review

Many algorithms for the determination of Nash equilibria require convexity and smoothness assumptions on the functions θ_ν. Prominent solution approaches comprise

- the Karush-Kuhn-Tucker approach and
- the variational inequality approach.

In contrast, the

- Nikaido-Isoda approach

neither needs convexity nor differentiability. However, even smooth convex θ_ν's lead to a nonsmooth nonconvex minimization problem.

F. Facchinei, C. Kanzow, *Generalized Nash equilibrium problems*, 4OR, 5 (2007), 173–210.

Literature review

So far spatial branch-and-bound methods for nonconvex continuous NEPs have not been suggested. For discrete NEPs

(aka *integer programming games*,
M. Carvalho, G. Dragotto, A. Lodi, S. Sankaranarayanan, *Integer Programming Games: A Gentle Computational Overview*, INFORMS TutORials in Operations Research, to appear)

branch-and-prune (but not -bound) has been studied in

S. Sagratella, *Computing all solutions of Nash equilibrium problems with discrete strategy sets*, SIOPT 26 (2016), 2190–2218

S. Schwarze, O. Stein, *A branch-and-prune algorithm for discrete Nash equilibrium problems*, COAP 86 (2023), 491–519.

Our nonconvex problem class

In our approach,

- strategy sets are boxes $\Omega_\nu \subseteq \mathbb{R}^{n_\nu}$ and, thus, convex,
- the players' objective functions $\theta_\nu : \Omega \to \mathbb{R}$ are continuous, but not assumed to be convex (neither in x nor in x^ν),
- the whole set E of Nash equilibria is approximated, not just a single equilibrium.

Bounding procedures

We require the availability of some convergent lower bounding procedure, i.e. for a lower semi-continuous function f and a box $X \subseteq \Omega$ we can compute a lower bound

$$\ell_f(X) \;\leq\; \min_{x \in X} f(x)$$

such that

$$\lim_{k\to\infty} \ell_f(X^k) \;=\; \lim_{k\to\infty} \min_{x \in X^k} f(x)$$

holds for any exhaustive sequence of boxes $(X^k)_{k\in\mathbb{N}}$.

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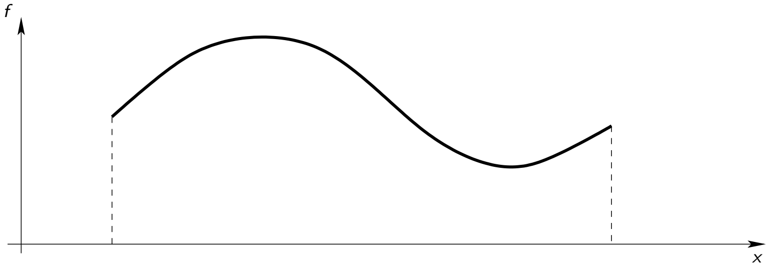
Convergent upper bounding procedures are defined analogously.

Bounding procedures



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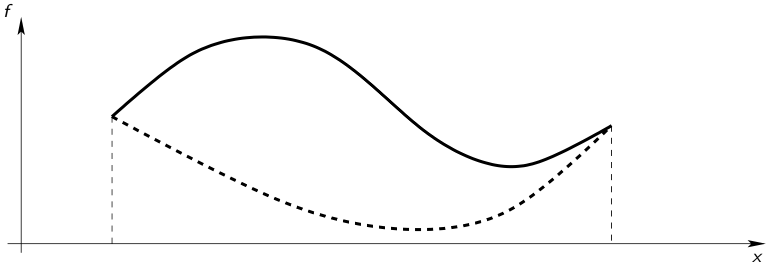

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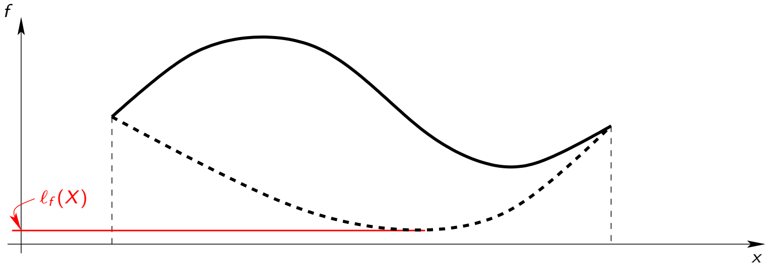

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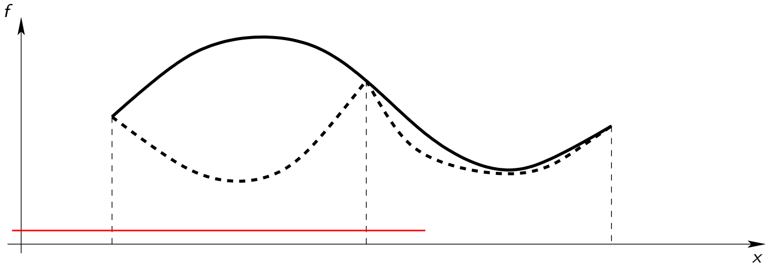
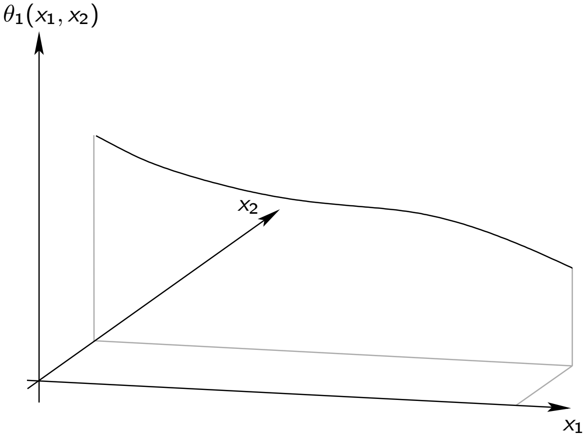


Main idea



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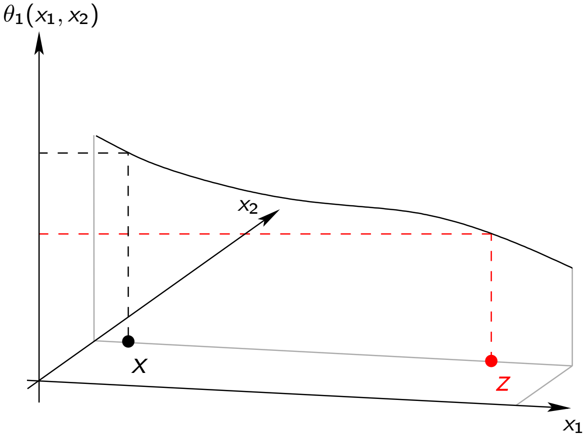

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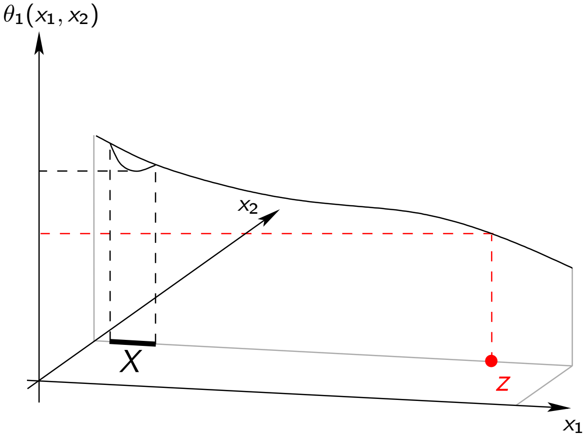

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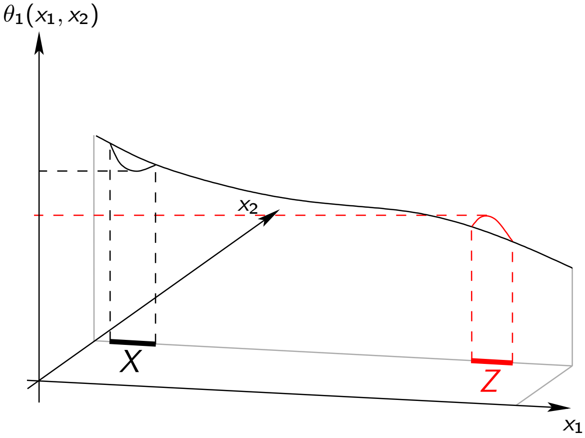

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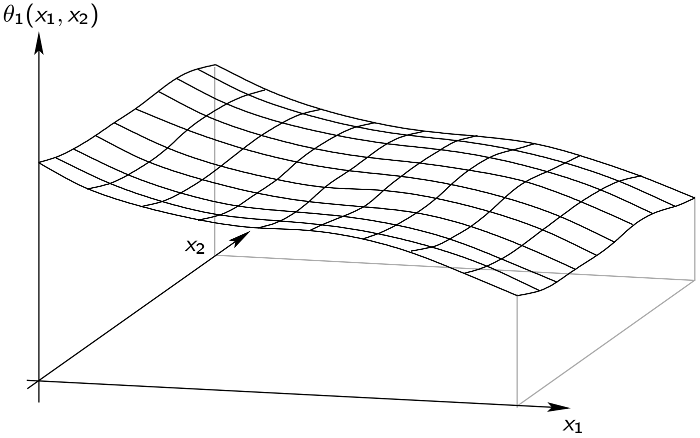

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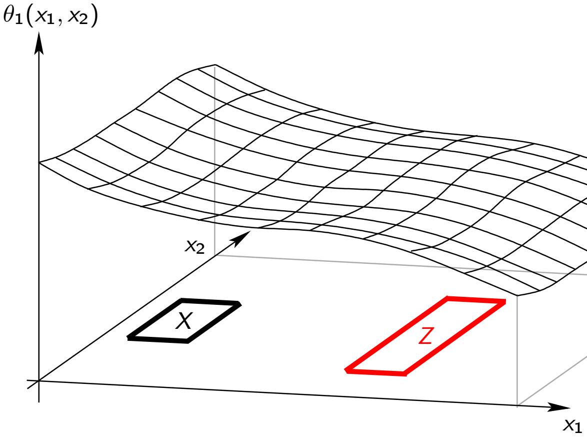

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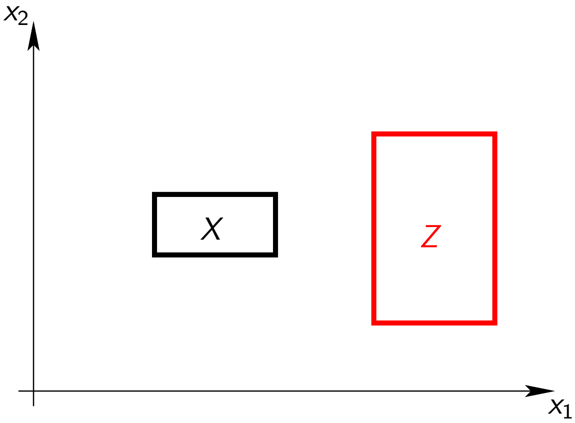

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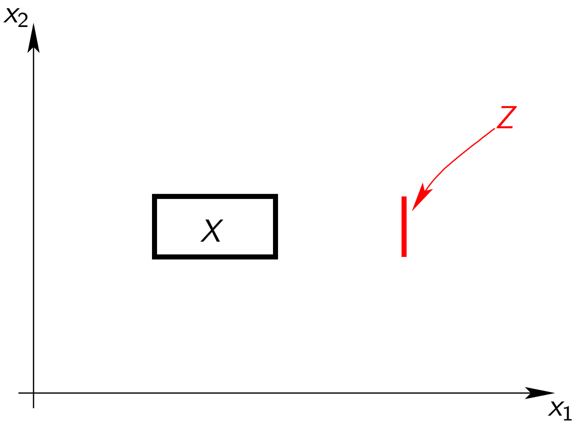

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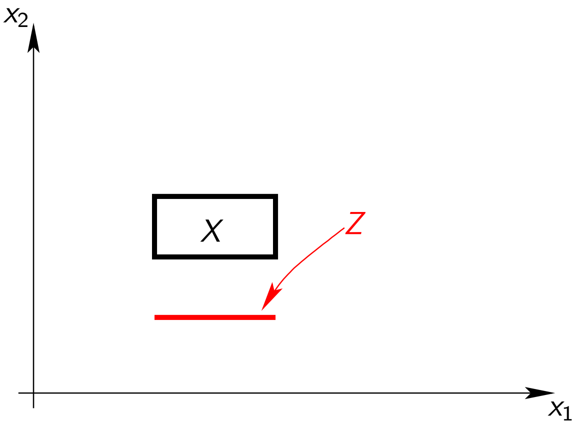

A discarding criterion

Proposition (Kirst/Schwarze/St. 2024)

Given: boxes $X, Z \subseteq \Omega$.
If there is at least one player ν *with*

- $\mathrm{pr}_{x^{-\nu}} X \subseteq \mathrm{pr}_{x^{-\nu}} Z$ *and*
- $\ell_{\theta_\nu}(X) > u_{\theta_\nu}(Z)$,

then X *does not contain any Nash equilibria.*

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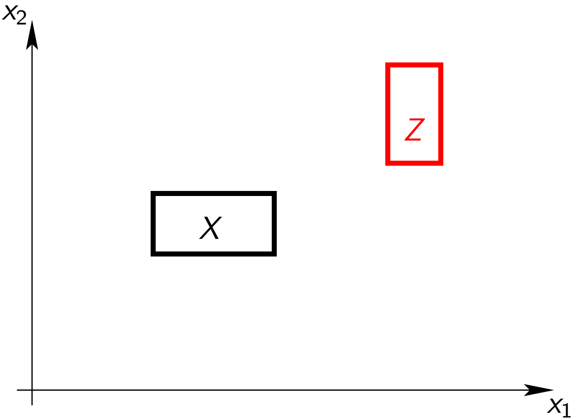
Main question no. 1: How to construct suitable boxes Z?

A discarding criterion



Unlike in B&B for global optimization, the source of Z cannot be (near) equilibrium points or boxes containing them.

Branch-and-bound algorithm

Algorithm 1: Branch-and-bound algorithm for nonconvex box-constrained NEPs

Initialization: Put list $\mathcal{L} := \{\Omega\}$, list $\mathcal{N} := \{\Omega\}$;

while $\exists X' \in \mathcal{N}$ *with* $\mathrm{diag}(X') > \tau$ **do**

  Step 1: Choose largest box $X \in \mathcal{N}$ and remove it from $\mathcal{N}$;

  Step 2: Divide X into X^1 and X^2 and append them to $\mathcal{N}$;

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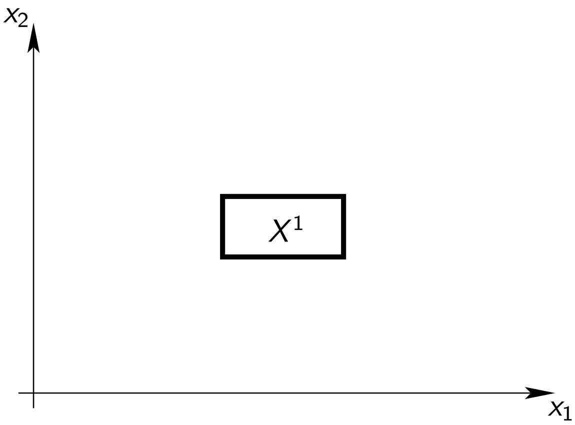
Illustration of discarding step



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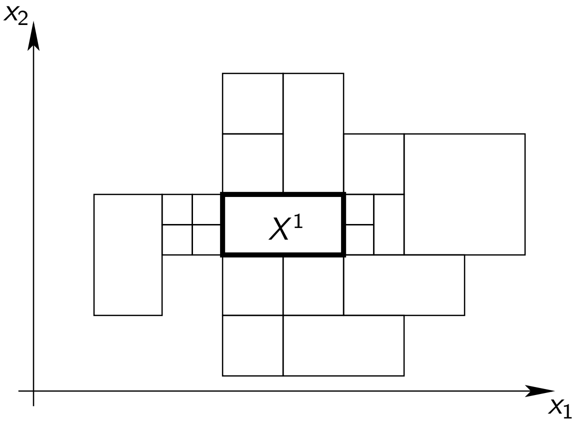

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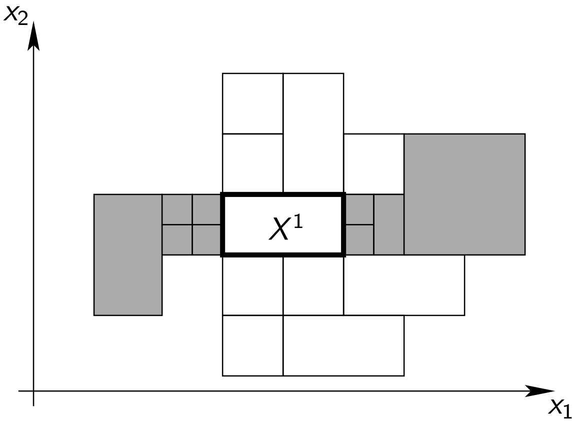


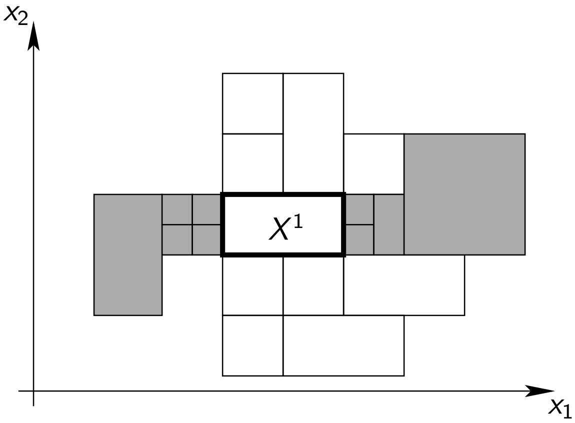
Definition of the sub-lists

In every iteration and for every player ν we consider the sub-lists

$$\mathcal{L}_\nu(X^1) \ := \ \left\{ Y \in \mathcal{L} \mid \left(\mathrm{pr}_{x^{-\nu}} Y \right) \cap \left(\mathrm{pr}_{x^{-\nu}} X^1 \right) \neq \emptyset \right\}$$

of $\mathcal{L}$ comprised of boxes that are of interest for player ν, since they may contain points which unilaterally improve points $x \in X^1$ in the player variable x^ν.

Illustration of discarding step



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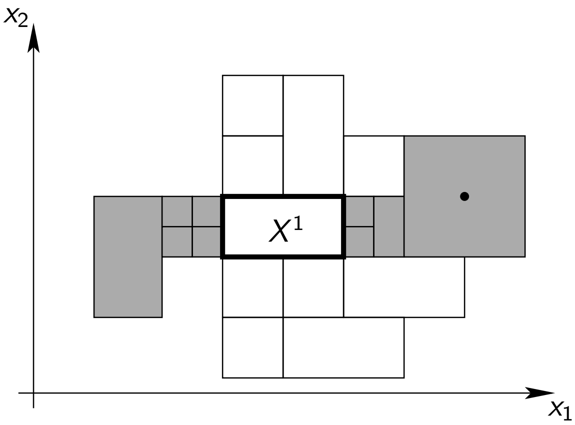

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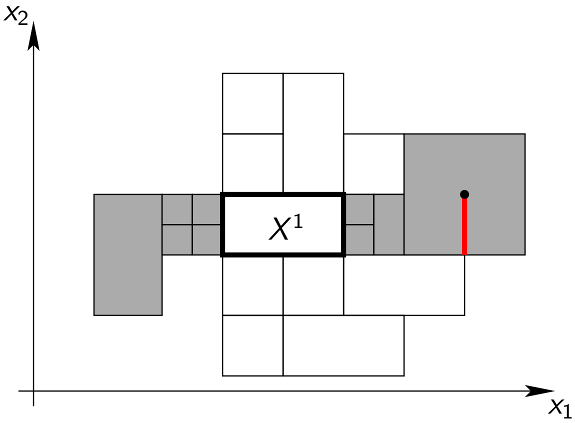

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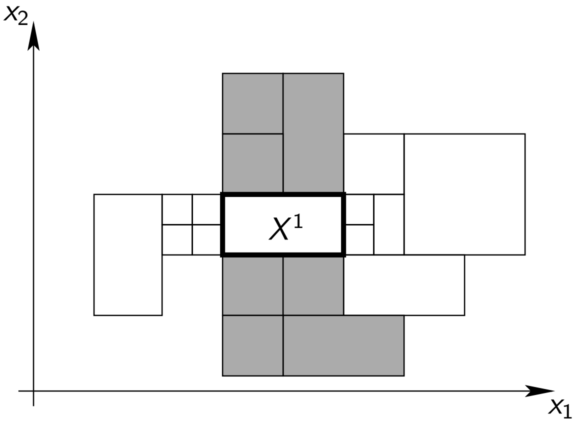

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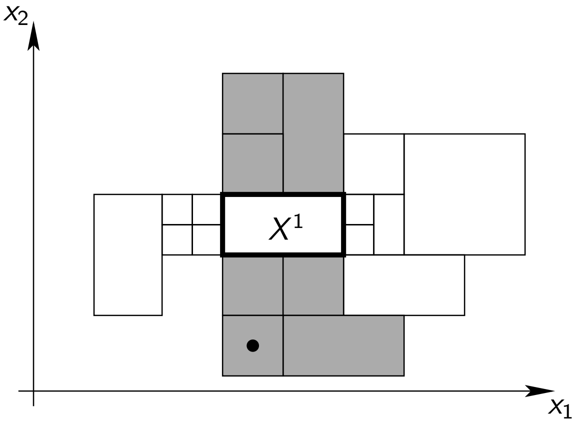

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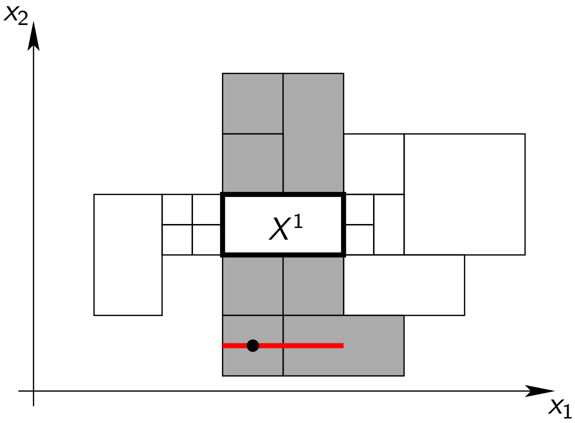

Branch-and-bound algorithm

Algorithm 2: Step 3 of the branch-and-bound algorithm for non-convex box-constrained NEPs

Step 3a: Try to discard box X^1:

for $\nu = 1, \ldots, N$ **do**

  Find $Y^\nu \in \mathcal{L}_\nu(X^1)$ with $\ell_{\theta_\nu}(Y^\nu) = \min_{Y \in \mathcal{L}_\nu(X^1)} \ell_{\theta_\nu}(Y)$;

  With midpoint $(\widehat{y}^1, \ldots, \widehat{y}^N)$ of Y^ν put

   $Z^\nu := X_1^1 \times \ldots \times [\widehat{y}^\nu, \widehat{y}^\nu] \times \ldots \times X_N^1$;

  if $\ell_{\theta_\nu}(X^1) > u_{\theta_\nu}(Z^\nu)$ **then**

    Remove X^1 from list $\mathcal{N}$;

  end

end

Step 3b: Proceed analogously for box X^2;

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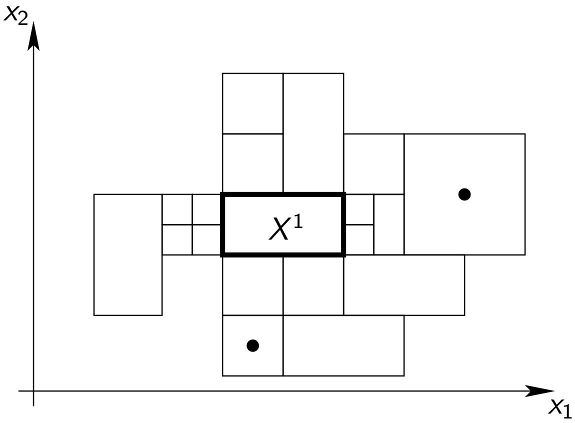
Improve boxes in list $\mathcal{L}$



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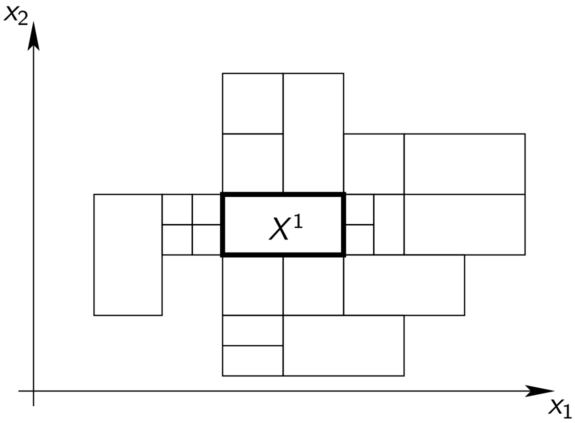

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If in Step 2 the box X is halved along a longest edge, then Algorithm 1 terminates after finitely many steps with

$$E \subseteq \bigcup_{X \in \mathcal{N}} X.$$

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Main question no. 2: How good is this approximation of E?

Convergence property of the algorithm

Theorem (Kirst/Schwarze/St. 2024)

Given a convergent lower bounding procedure, consider the infinite branch-and-bound sequence generated by Algorithm 1 for $\tau = 0$ *and put*

$$E^k := \bigcup_{X \in \mathcal{N}_k} X,$$

with $\mathcal{N}_k$ *denoting the list* $\mathcal{N}$ *in iteration* k*.*

Then with the Hausdorff distance δ *we have* $\lim_{k\to\infty} \delta(E^k, E) = 0$*.*

Inclusion of certain fathoming steps for $\mathcal{L}$ is possible as well, but omitted here for ease of presentation.

Convergence property of the algorithm

Unfortunately, using Algorithm 1 with $\tau > 0$ does not yield

$$\delta(E^k, E) \leq \tau$$

for the final iterate k, but we simply stop with

$$\max_{X \in \mathcal{N}_k} \operatorname{diag}(X) \leq \tau,$$

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Main question no. 3: How to control the approximation quality?

The notion of ε-Nash equilibria

An ε-Nash equilibrium is a point $\bar{x} \in \Omega$ such that for all ν:

$$\theta_\nu(\bar{x}^\nu, \bar{x}^{-\nu}) \leq \theta_\nu(x^\nu, \bar{x}^{-\nu}) + \varepsilon \quad \text{for all} \quad x^\nu \in \Omega_\nu\,.$$

This means that for each ν the point $\bar{x}^\nu$ is an ε-minimal point of

$$P_\nu(\bar{x}^{-\nu}): \quad \min_{x^\nu} \; \theta_\nu(x^\nu, \bar{x}^{-\nu}) \quad \text{s.t.} \quad x^\nu \in \Omega_\nu.$$

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For the following we assume $\delta\,(E_\varepsilon^<, E_\varepsilon) = 0$.

Inner approximation of $E_\varepsilon^<$

We have

$$\max_{\nu}\left(u_{\theta_\nu}(X^1) - \min_{Y\in\mathcal{L}_\nu(X^1)} \ell_{\theta_\nu}(Y)\right) < \varepsilon \quad \Rightarrow \quad X^1 \subseteq E_\varepsilon^<$$

because all $x \in X^1$ satisfy for all ν

$$\theta_\nu(x^\nu, x^{-\nu}) \leq u_{\theta_\nu}(X^1) < \min_{Y\in\mathcal{L}_\nu(X^1)} \ell_{\theta_\nu}(Y) + \varepsilon \leq \min_{y^\nu\in\Omega_\nu} \theta_\nu(y^\nu, x^{-\nu}) + \varepsilon.$$

Inner approximation of $E_\varepsilon^<$

We have

$$\max_\nu \left(u_{\theta_\nu}(X^1) - \min_{Y \in \mathcal{L}_\nu(X^1)} \ell_{\theta_\nu}(Y) \right) < \varepsilon \quad \Rightarrow \quad X^1 \subseteq E_\varepsilon^<$$

because all $x \in X^1$ satisfy for all ν

$$\theta_\nu(x^\nu, x^{-\nu}) \leq u_{\theta_\nu}(X^1) < \min_{Y \in \mathcal{L}_\nu(X^1)} \ell_{\theta_\nu}(Y) + \varepsilon \leq \min_{y^\nu \in \Omega_\nu} \theta_\nu(y^\nu, x^{-\nu}) + \varepsilon.$$

We collect such boxes in a list $\widetilde{\mathcal{N}}$ and obtain the chain of inclusions

$$\bigcup_{\widetilde{X} \in \widetilde{\mathcal{N}}} \widetilde{X} \subseteq E_\varepsilon^< \subseteq E_\varepsilon \subseteq \bigcup_{X \in \mathcal{N}} X.$$

Termination criterion

With an approximation tolerance $\tau > 0$ we wish to terminate for

$$\delta\left(\bigcup_{\widetilde{X}\in\widetilde{\mathcal{N}}} \widetilde{X}, \bigcup_{X\in\mathcal{N}} X\right) \;\leq\; \tau.$$

Due to

$$\delta\left(\bigcup_{\widetilde{X}\in\widetilde{\mathcal{N}}} \widetilde{X}, \bigcup_{X\in\mathcal{N}} X\right) \;\leq\; \max_{X\in\mathcal{N}} \min_{\widetilde{X}\in\widetilde{\mathcal{N}}} \|\Delta(X,\widetilde{X})\|_2$$

with

$$\Delta_i([\underline{a},\overline{a}],[\underline{b},\overline{b}]) \;:=\; \max\{0, \underline{b}_i - \underline{a}_i, \overline{a}_i - \overline{b}_i\}, \quad i = 1,\ldots,n\,,$$

this follows from the tractable termination criterion

$$\max_{X\in\mathcal{N}} \min_{\widetilde{X}\in\widetilde{\mathcal{N}}} \|\Delta(X,\widetilde{X})\|_2 \;\leq\; \tau.$$

Modified algorithm

These considerations lead to modifications of Algorithm 1 concerning

- the computation of strict and nonstrict ε-Nash equilibria,
- the maintenance of the additional list $\widetilde{\mathcal{N}}$,
- the more appropriate termination criterion.

Convergence properties of the modified algorithm

Theorem (Kirst/Schwarze/St. 2024)

Let a convergent lower bounding procedure and $\varepsilon > 0$ be given.

a) *For $E_\varepsilon^< \neq \emptyset$ and $\delta(E_\varepsilon, E_\varepsilon^<) = 0$ the modified algorithm with $\tau > 0$ terminates after a finite number of iterations with $\mathcal{N} \neq \emptyset$.*

b) *For $E_\varepsilon = \emptyset$ the modified algorithm with $\tau > 0$ terminates after a finite number of iterations with $\mathcal{N} = \emptyset$.*

Illustrative examples

Simple implementation:

- Python 3.10.8
- standard computer (Intel i7 processor, 3.60 GHz, 32 GB of RAM)
- lower bounding procedures based on centered forms (Krawczyk/Nickel 1982)

Four examples are tested:

- two players with unique equilibrium
- two players with multiple equilibria
- two players with no equilibrium
- a three dimensional instance

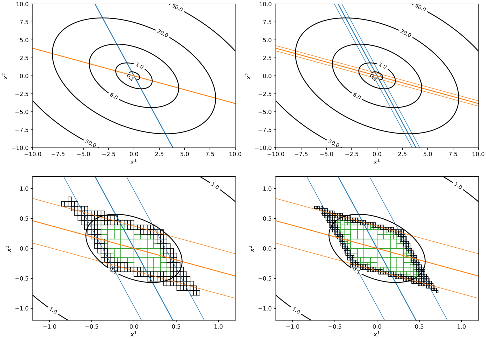
Example from Krawczyk/Uryasev 2000

Objective functions:

$$\theta_1(x^1, x^2) = \theta_2(x^1, x^2) = \frac{(x^1 + x^2)^2}{4} + \frac{(x^1 - x^2)^2}{9}.$$

Strategy sets: $\Omega_1 = \Omega_2 = [-10, 10]$.

Problem taken from Krawczyk/Uryasev 2000



$\tau = 0.1$ $\tau = 0.05$

Approximations of $E_{0.05}^{<}$ and $E_{0.05}$ by uniformly refining $\mathcal{N} \setminus \widetilde{\mathcal{N}}$

Example based on Beck/St. 2024

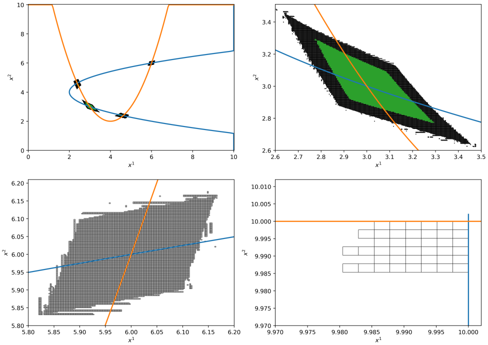
Objective functions:

$$\begin{aligned}\theta_1(x^1, x^2) &= \frac{(x^1)^2}{2} - q(x^2) \cdot x^1, \\ \theta_2(x^1, x^2) &= \frac{(x^2)^2}{2} - q(x^1) \cdot x^2\end{aligned}$$

with $q(x) = (x-4)^2 + 2$.

Strategy sets: $\Omega_1 = \Omega_2 = [0, 10]$.

Example based on Beck/St. 2023



$\tau = 0.005$

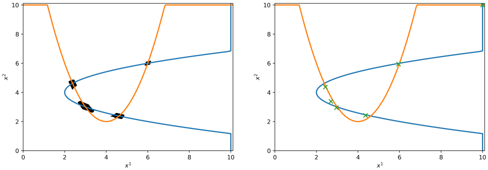
Approximation of $E^{<}_{0.05}$ and $E_{0.05}$ by uniformly refining $\mathcal{N} \setminus \widetilde{\mathcal{N}}$

Example based on Beck/St. 2023

τ	k	$\lvert\mathcal{N} \setminus \widetilde{\mathcal{N}}\rvert$	$\lvert\widetilde{\mathcal{N}}\rvert$	$\lvert\mathcal{L}\rvert$
0.05	7,145	3,355	0	22,837
0.02	22,872	11,155	0	73,201
0.01	52,782	28,847	709	174,794
0.005	129,132	71,376	11,810	444,998

Approximation of $E^{<}_{0.05}$ and $E_{0.05}$ by uniformly refining $\mathcal{N} \setminus \widetilde{\mathcal{N}}$

Example based on Beck/St. 2023



$\tau = 0.5$

Approximation of $E^{<}_{0.05}$ and $E_{0.05}$ by modified algorithm

$$\left(\; k = 121{,}520, \quad |\mathcal{N} \setminus \widetilde{\mathcal{N}}| = 64{,}194, \quad |\widetilde{\mathcal{N}}| = 6 \; \right)$$

Classical example in economics

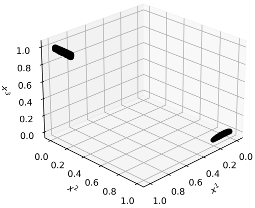
(inspired by an economical situation, see Sagratella 2017, Ex. 1,2)

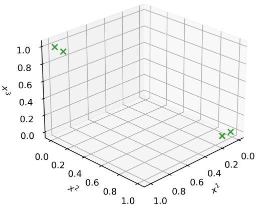
Objective functions:

$$\theta_1(x^1, x^2) = \frac{(x_1^1)^2}{2} + \frac{(x_2^1)^2}{2} + x_1^1 x_2^1 - x_1^1 x_1^2 - x_1^1 - x_2^1 \,,$$
$$\theta_2(x^1, x^2) = \frac{(x_1^2)^2}{2} + x_2^1 x_1^2 - x_1^2 \,.$$

Strategy sets: $\Omega_1 = [0,1]^2$, $\Omega_2 = [0,1]$.

Classical example in economics





$\tau = 0.1$

Approximation of $E^{<}_{0.01}$ and $E_{0.01}$ by modified algorithm

$$(\ k = 51{,}220, \quad |\mathcal{N} \setminus \widetilde{\mathcal{N}}| = 4{,}377, \quad |\widetilde{\mathcal{N}}| = 5 \ )$$

Final remarks

- Implementation is rather simple.
- Optional fathoming steps for $\mathcal{L}$ are included (Kirst/Schwarze/St. 2024), but not discussed here.
- Numerical results are so far only proof of concept, but method is to be tested on real-world applications (e.g. at WUR).
- Generalization to more complicated constraints and GNEPs is nontrivial and subject of our current research.

References

[1] M. Beck, O. Stein, *Semi-infinite models for equilibrium selection*, Minimax Theory and its Applications, Minimax Theory and its Applications, Vol. 9 (2024), 1-18.

[2] P. Kirst, S. Schwarze, O. Stein, *A branch-and-bound algorithm for non-convex Nash equilibrium problems*, SIAM Journal on Optimization, to appear.

[3] R. Krawczyk, K. Nickel, *Die zentrische Form in der Intervallarithmetik, ihre quadratische Konvergenz und ihre Inklusionsisotonie*, Computing, 28 (1982), 117-137.

[4] J.B. Krawczyk, S. Uryasev, *Relaxation algorithms to find Nash equilibria with economic applications*, Environmental Modeling & Assessment, 5 (2000), 63-73.

[5] S. Sagratella, *Computing equilibria of Cournot oligopoly models with mixed-integer quantities*, Mathematical Methods of Operations Research, 86 (2017), 549–565.