Addressing hierarchical jointly-convex generalized Nash equilibrium problems with nonsmooth payoffs

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Outline

motivating application: hierarchical multi-portfolio selection

generalized variational inequality formulations

projected gradient-like algorithms

numerical results

multi-portfolio selection

$\nu = 1, \ldots, N:$

$$\text{minimize}_{y^\nu}\ \theta^I_\nu(y^\nu, y^{-\nu}) + \varphi^I_\nu(y^\nu) \quad \text{s.t.} \quad y^\nu \in Y_\nu$$

$$\theta^I_\nu(y^\nu, y^{-\nu}) = -I_\nu(y^\nu) + \rho_\nu R_\nu(y^\nu) + TC_\nu(y^\nu, y^{-\nu})$$

$$\varphi^I_\nu(y^\nu) = \tau_\nu \|y^\nu\|_1$$

$$Y_\nu = \left\{ y^\nu \in [l_\nu, u_\nu]^K : \sum_{i=1}^{K} y_i^\nu \leq 1 \right\}$$

Yang, Y., Rubio, F., Scutari, G., Palomar, D. P. (2013). Multi-portfolio optimization: A potential game approach. IEEE Tran Sign Proc, 61

Lampariello, L., Neumann, C., Ricci, J. M., Sagratella, S., Stein, O. (2021). Equilibrium selection for multi-portfolio optimization. Europ J Oper Res, 295

Nash equilibrium problem (lower-level NEP)

find $y \in Y \triangleq Y_1 \times \cdots \times Y_N$ such that

$$\nu = 1, \ldots, N :$$
$$\theta_\nu^l(y^\nu, y^{-\nu}) + \varphi_\nu^l(y^\nu) \leq \theta_\nu^l(v^\nu, y^{-\nu}) + \varphi_\nu^l(v^\nu), \; \forall v^\nu \in Y_\nu$$

assumptions A

A1 Y_ν is nonempty, convex and compact, $\nu = 1, \ldots, N$

A2 $\theta_\nu^l \in C^1$ is convex with respect to y^ν, $\nu = 1, \ldots, N$

A3 $\left[\nabla_{y^\nu} \theta_\nu^l\right]_{\nu=1}^N$ is monotone on Y

A4 φ_ν^l is convex and locally Lipschitz, $\nu = 1, \ldots, N$

hierarchical multi-portfolio selection

$\mu = 1, \ldots, M$:

$$\text{minimize}_{x^\mu}\, \theta^u_\mu(x^\mu, x^{-\mu}) + \varphi^u_\mu(x^\mu) \quad \text{s.t.} \quad (x^\mu, x^{-\mu}) \in E$$

$$\begin{aligned} x^\mu &= [y^\nu]_{\nu \in \mathcal{S}_\mu} \\ \theta^u_\mu(x^\mu, x^{-\mu}) &= -\sum_{\nu \in \mathcal{S}_\mu} I_\nu(y^\nu) + \rho_\mu \sum_{\nu \in \mathcal{S}_\mu} R_\nu(y^\nu) + TC_\mu(x^\mu, x^{-\mu}), \\ \varphi^u_\mu(x^\mu) &= \tau_\mu \sum_{\nu \in \mathcal{S}_\mu} \|y^\nu\|_1 \\ E &\triangleq \text{solutions of the lower-level NEP} \end{aligned}$$

Lampariello, L., Priori, G., Sagratella, S. (2022). On the solution of monotone nested variational inequalities. Math Meth Oper Res, 96

hierarchical generalized Nash equilibrium problem

find $x \in E$ such that

$\mu = 1, \ldots, M$:
$$\theta_\mu^u(x^\mu, x^{-\mu}) + \varphi_\mu^u(x^\mu) \leq \theta_\mu^u(w^\mu, x^{-\mu}) + \varphi_\mu^u(w^\mu), \ \forall w^\mu : (w^\mu, x^{-\mu}) \in E$$

assumptions B

B1 $\theta_\mu^u \in C^1$ is convex with respect to x^μ, $\mu = 1, \ldots, M$

B2 $\left[\nabla_{x^\mu} \theta_\mu^u\right]_{\mu=1}^M$ is monotone on Y

B3 φ_μ^u is convex and locally Lipschitz, $\mu = 1, \ldots, M$

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lower-level GVI formulation

find $y \in Y$ such that

$$\exists f_y \in F(y) : \quad f_y^T (v - y) \geq 0, \quad \forall v \in Y$$

where

$$F(y) = \left[\nabla_{y^\nu} \theta_\nu^l (y) \right]_{\nu=1}^N + \left[\partial_{y^\nu} \varphi_\nu^l (y^\nu) \right]_{\nu=1}^N$$

properties under assumptions A and B

- $E = SOL(F, Y)$
- $SOL(F, Y)$, and then E, are nonempty and compact
- F is maximal monotone and outer semi-continuous

$\Longrightarrow$ $SOL(F, Y)$, and then E, are convex sets

$\Longrightarrow$ the hierarchical GNEP is jointly-convex

on maximal monotonicity

- $T : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ is monotone if

$$(\widehat{t} - \widetilde{t})^T (\widehat{u} - \widetilde{u}) \geq 0, \;\; \forall \left(\widehat{u}, \widehat{t}\right), \left(\widetilde{u}, \widetilde{t}\right) \in \mathrm{gph}(T)$$

- $T : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$ is maximal monotone if it is monotone and

$$\forall \left(\widehat{u}, \widehat{t}\right) \notin \mathrm{gph}(T), \; \exists \left(\widetilde{u}, \widetilde{t}\right) \in \mathrm{gph}(T) \; : \; (\widehat{t} - \widetilde{t})^T (\widehat{u} - \widetilde{u}) < 0$$

- every continuous monotone single-valued mapping is maximal monotone (not true for multi-valued mappings)

Lemma

consider the following conditions on $T : \mathbb{R}^p \rightrightarrows \mathbb{R}^p$:

$$(\widehat{u}, \widehat{t}) \in gph(T) \tag{L1}$$

$$(\widehat{t} - t)^T (\widehat{u} - u) \geq 0, \;\; \forall (u, t) \in gph(T) \tag{L2}$$

if T is monotone, then (L1) implies (L2)
if T is maximal monotone, then (L2) implies (L1)

hierarchical GVI formulation

find $x \in SOL(F, Y)$ such that

$$\exists g_x \in G(x) : \quad g_x^T(w - x) \geq 0, \quad \forall w \in SOL(F, Y)$$

where

$$G(x) = \left[\nabla_{x^\mu} \theta_\mu^u(x)\right]_{\mu=1}^M + \left[\partial_{x^\mu} \varphi_\mu^u(x^\mu)\right]_{\mu=1}^M$$

properties under assumptions A and B

- every $x \in SOL(G, SOL(F, Y))$ is a (variational) solution of the hierarchical GNEP
- $SOL(G, SOL(F, Y))$ is nonempty and compact

  $\Longrightarrow$ the solution set of the hierarchical GNEP is nonempty
- G is maximal monotone and outer semi-continuous

  $\Longrightarrow$ $SOL(G, SOL(F, Y))$ is convex

Tikhonov GVI subproblem

given $\eta \geq 0$, find $y \in Y$ such that

$$\exists h_y^\eta \in H_\eta(y): \quad {h_y^\eta}^T (v - y) \geq 0, \quad \forall v \in Y$$

where

$$H_\eta(y) = F(y) + \eta G(y)$$

Theorem

given $\delta \geq 0$, let $y \in Y$ such that

$$\exists h_y^\eta \in H_\eta(y): \quad {h_y^\eta}^T (v - y) \geq -\delta, \quad \forall v \in Y$$

then

$$\exists f_y \in F(y): \quad f_y^T (v - y) \geq -\delta - \eta\Lambda, \quad \forall v \in Y$$

$$\exists g_y \in G(y): \quad g_y^T (v - y) \geq -\delta/\eta, \quad \forall v \in SOL(F, Y)$$

where $\Lambda = \max_{y \in Y} \max_{g_y \in G(y)} \|g_y\| \max_{x,v \in Y} \|x - v\|$

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projected gradient-like scheme

given $\{\gamma_k\}$, $\{\eta_k\}$, $y_1 \in Y$
for every $k = 1, \ldots$ compute:

$$f_{y_k} \in F(y_k), \quad g_{y_k} \in G(y_k), \quad h_{y_k}^{\eta_k} \leftarrow f_{y_k} + \eta_k g_{y_k},$$
$$y_{k+1} \leftarrow P_Y(y_k - \gamma_k h_{y_k}^{\eta_k})$$

for every $k = \overline{k}, \ldots$ compute:

$$z_k \leftarrow \frac{\sum_{j=\overline{k}}^{k} \gamma_j y_j}{\sum_{j=\overline{k}}^{k} \gamma_j}$$

convergence properties 1

assumptions C

C1 $\{\gamma_k\}$ is non-increasing, $0 < \gamma_k \to 0$, and $\{\gamma_k\} \notin \ell_1$

C2 $\{\eta_k\}$ is non-increasing, $0 < \eta_k \to \eta \geq 0$

Theorem

a) *if* $\{\gamma_k\} \in \ell^2$, *and* $\{\gamma_k(\eta_j - \eta)\} \in \ell^1$, *then*
$\forall\, u \in SOL(H_\eta, Y), \exists l_u \geq 0 \,:\, \lim_{k\to\infty} \|y_k - u\|^2 = l_u$

b) *if* $y_k \to \overline{y}$, *then* $\overline{y} \in SOL(H_\eta, Y)$

c) $\|y_{k+1} - y_k\| \to 0$

d) $z_k \to \overline{z} \in SOL(H_\eta, Y)$

convergence properties 2

additional assumptions D

D1 $\eta = 0$

D2 $\frac{\gamma_k}{\eta_k} \to 0$

D3 $\eta_k \sum_{j=1}^{k} \gamma_j \to \infty$

Theorem

a) *if* $y_k \to \overline{y}$, *then* $\overline{y} \in SOL(G, SOL(F, Y))$

b) $z_k \to \overline{z} \in SOL(G, SOL(F, Y))$

complexity bounds

consider harmonic sequences:

$$\gamma_k = \frac{\overline{\gamma}}{k^\alpha}, \quad \eta_k = \frac{\overline{\eta}}{k^\beta} + \eta, \quad k \geq 1$$

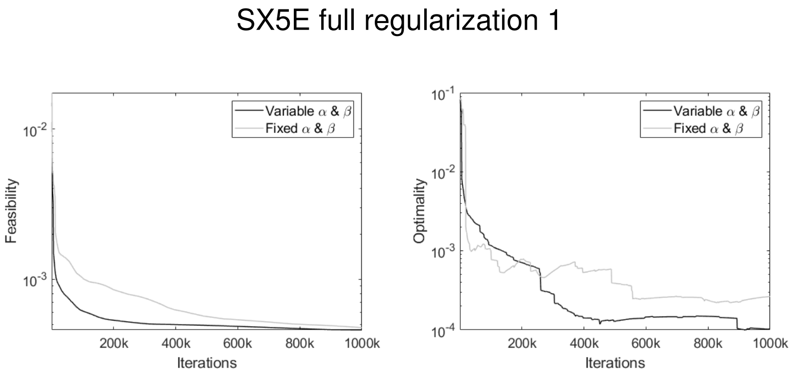
α	β	convergence properties	complexity
$1-\epsilon$	$1-\epsilon$	$\limsup_{\Delta\to\infty} \Vert y_{k+\Delta} - u\Vert^2 - \Vert y_k - u\Vert^2 \leq \delta,$ $u \in \text{SOL}(H_\eta, Y)$	$\mathcal{O}(\delta^{-1/(1-2\epsilon)})$
0.5	0.5	$h_v^{\eta\,T}(v - z_k) \geq -\delta,\ \forall v \in Y,\ h_v^\eta \in H_\eta(v)$	$\mathcal{O}(\delta^{-2})$
0.5	0.25	$f_v^T(v - z_k) \geq -\delta,\ \forall v \in Y,\ f_v \in F(v)$ $g_v^T(v - z_k) \geq -\delta,\ \forall v \in \text{SOL}(F, Y),\ g_v \in G(v)$	$\mathcal{O}(\delta^{-4})$

Outline

hierarchical multi-portfolio selection models

- $N = 25$, $M = 5$, $\mathcal{S}_\mu = \{1 + 5(\mu - 1), \ldots, 5 + 5(\mu - 1)\}$
- data sets: $K = 10$ assets from Euro Stoxx 50 (SX5E); $K = 29$ assets from Dow Jones Industrial Average (DJIA)
- accounts 1-15 are regularized at the lower-level; accounts 16-25 are regularized at the upper-level
- $\text{feas}(y, f_y) \triangleq \max_{v \in Y} -f_y^T(v - y) \geq -\min_{v \in Y} \min_{f_v \in F(v)} f_v^T(v - y)$
- $\text{opt}(y, f_y, g_y) \triangleq \max_{v \in Y} -(1/\eta)(f_y + \eta g_y)^T(v - y) \geq -\min_{v \in \text{SOL}(F,Y)} \min_{g_v \in G(v)} g_v^T(v - y)$

results 1


SX5E full regularization 1
Feasibility
Iterations
Variable α & β
Fixed α & β
10^{-2}
10^{-3}
200k
400k
600k
800k
1000k
Optimality
Iterations
10^{-1}
10^{-2}
10^{-3}
10^{-4}
Variable α & β
Fixed α & β
200k
400k
600k
800k
1000k

results 2

			feas$(z_{\bar{l}}, f_{z_{\bar{l}}})$		feas$(y_{\bar{l}}, f_{y_{\bar{l}}})$	opt$(y_{\bar{l}}, f_{y_{\bar{l}}}, g_{y_{\bar{l}}})$
		$\bar{k} = 0$	$\bar{k} = 0.4\bar{l}$	$\bar{k} = 0.8\bar{l}$		
SX5E	No reg	2.2225e-04	9.8603e-05	9.6364e-05	9.5698e-05	9.1130e-05
	Low. reg. 1	1.0859e-03	2.3361e-04	2.2854e-04	2.2719e-04	8.6980e-05
	Low. reg. 2	2.4301e-03	3.1726e-04	3.0886e-04	3.0703e-04	6.1042e-05
	Full reg. 1	1.5472e-03	4.7882e-04	4.6261e-04	4.5935e-04	1.0150e-04
	Full reg. 2	2.8817e-03	5.5899e-04	5.4486e-04	5.4262e-04	6.9896e-05
DIJA	No reg.	9.2238e-04	3.2620e-04	3.1930e-04	3.1754e-04	1.9082e-04
	Low. reg. 1	3.4512e-03	6.7104e-04	6.7104e-04	6.5528e-04	1.7030e-04
	Low. reg. 2	8.2720e-04	8.2720e-04	8.3136e-04	8.2720e-04	2.0577e-04
	Full reg. 1	3.8326e-03	8.8701e-04	8.6774e-04	8.6284e-04	2.6538e-04
	Full reg. 2	7.6527e-03	1.0600e-03	1.0416e-03	1.0365e-03	2.7482e-04

	SX5E		DIJA	
#Accounts	1–15	16–25	1–15	16–25
No regularization	0.00%	0.00%	0.00%	0.00%
Lower regularization 1	28.00%	0.00%	38.62%	0.00%
Lower regularization 2	44.67%	0.00%	56.78%	0.00%
Full regularization 1	28.00%	25.00%	38.62%	12.07%
Full regularization 2	44.67%	24.00%	56.55%	11.72%